Parametric Representation of the Resonance Set of Polynomial

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Abstract. We consider the resonance set of a real polynomial, i. e. the set of all the points of the coefficient space at which the polynomial has commensurable zeroes. The constructive algorithm of computation of polynomial representation of the resonance set is provided. The structure of the resonance set of a polynomial of degree n is described in terms of partitions of number n. The main algorithms, described in the preprint, are organized as a library of the computer algebra system `Maple`.

Introduction

Let $f_n(x)$ be a monic polynomial of degree n with real coefficients

$$f_n(x) \stackrel{\text{def}}{=} x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n. \tag{1}$$

The n-dimensional space $\Pi \equiv \mathbb{R}^n$ of its coefficients $a_1, a_2, \dots a_n$ is called the *coefficient space* of polynomial (1).

Definition 1. A pair of roots t_i, t_j, $i, j = 1, \dots, n$, $i \neq j$, of the polynomial (1) is called $p:q$-*commensurable* if $t_i : t_j = p : q$.

Here and further we consider that $p \in \mathbb{Z}\backslash\{0\}$, $q \in \mathbb{N}$, i.e. we exclude the case when one of the commensurable root t_i or t_j is equal to zero due to the fact that zero root is commensurable with any other root.

Definition 2. Resonance set $\mathcal{R}_{p:q}(f_n)$ of the polynomial $f_n(x)$ is called the set of all points of the coefficient space Π at which $f_n(x)$ has at least a pair of $p:q$-commensurable roots, i.e.

$$\mathcal{R}_{p:q}(f_n) = \{P \in \Pi : \exists i, j = 1, \dots, n, t_i : t_j = p : q\}. \tag{2}$$

The aim of this work is to present an algorithm of constructing polynomial representation of the resonance set $\mathcal{R}_{p:q}(f_n)$ of the real polynomial $f_n(x)$.

1. Condition on $p:q$-commensurability of polynomial roots

Let polynomial (1) has a pair of $p:q$-commensurable roots. It means that two polynomials $f_n(px)$ and $f_n(qx)$ has common root, or in terms of resultant one has that $\mathrm{Res}_x(f_n(px), f_n(qx)) = 0$. In the case when $p = q$ both polynomials $f_n(px)$ and $f_n(qx)$ have exactly n common roots. In case $a_n = 0$ one of the root is equal to zero, therefore resultant can be written in the form

$$\mathrm{Res}_x(f_n(px), f_n(qx)) = a_n(p-q)^n \,\mathrm{GD}_{p:q}(f_n), \tag{3}$$

where $\mathrm{GD}_{p:q}(f_n)$ is so called *generalized discriminant* of the polynomial (1) introduced in [1].

Polynomial (1) may have some pairs of $p:q$-commensurable roots.

Definition 3. The *chain* $\mathrm{Ch}^{(k)}_{p:q}(t_i)$ *of* $p:q$*-commensurable roots* of length k (shortly *chain of roots*) is called the finite part of geometric progression with common ratio p/q and scale factor t_i, each member of which is a root of polynomial (1). The value t_i is called the *generating root.*

The detail structure of the resonance set (2) can be described with the help of so called *i-th generalized subdiscriminants* $\mathrm{GD}^{(i)}_{p:q}(f_n)$, which are nontrivial factors of i-th subresultants of pair of polynomials $f_n(px)$ and $f_n(qx)$. Such subresultants can be computed as i-th inners of Sylvester matrix constructed from the coefficients of mentioned above polynomials. For more details see [2].

Theorem 1. *Polynomial $f_n(x)$ has exactly $n-d$ different chains of roots $\mathrm{Ch}^{(i)}_{p:q}(t_j)$, $j = 1, \dots, n-d$ if and only if in the sequence $\left\{\mathrm{GD}^{(i)}_{p:q}(f_n), i = 0, \dots, n-1\right\}$ of i-th generalized subdiscriminants of $f_n(x)$ the first nonzero subdiscriminant is d-th generalized subdiscriminant $\mathrm{GD}^{(d)}_{p:q}(f_n)$.*

2. Parametrization of the resonance set

Consider a partition $\lambda = [1^{n_1}2^{n_2}\dots i^{n_i}\dots]$ of $n \in \mathbb{N}$. Functions $p(n)$ and $p_l(n)$ return the number of all partitions and the number of all partitions of the length l of natural number n respectively. The value i in the partition λ defines the length of chain $\mathrm{Ch}^{(i)}_{p:q}(t_i)$ for a corresponding generating root t_i, the value n_i defines the number of different generating roots, which give the chains of root of the length i. Then $l = \sum_i n_i$ is the number of different generating roots of the polynomial $f_n(x)$ for the certain coefficient of commensurability $p:q$, and $\sum_i in_i = n$.

Any partition λ of degree n of polynomial (1) defines a certain structure of $p:q$-commensurable roots of this polynomial and it corresponds to some algebraic variety $\mathcal{V}^i_l$, $i = 1, \dots, p_l(n)$ of dimension l in the coefficient space Π. The number of such varieties of dimension l is equal to $p_l(n)$ and total number of all varieties consisting the resonance set $\mathcal{R}_{p:q}(f_n)$ is equal to $p(n) - 1$. It is so because the partition $[1^n]$ corresponds to the case when all the n roots of polynomial (1) are not commensurable.

Algorithm for parametric representation of any variety $\mathcal{V}_l$ from the resonance set $\mathcal{R}_{p:q}(f_n)$ is based on the following

Theorem 2. *Let* $\mathcal{V}_l$, $\dim \mathcal{V}_l = l$, *be a variety on which polynomial* (1) *has* l *different chains of* $p : q$*-commensurable roots and the chain generated by the root* t_1 *has length* $m > 1$. *Let denote by* $\mathbf{r}_l(t_1, t_2, \ldots, t_l)$ *parametrization of variety* $\mathcal{V}_l$. *Therefore the following formula*

$$\mathbf{r}_l(t_1, \ldots, t_l, v) = \mathbf{r}_l(t_1, \ldots, t_l) + \frac{p(q^m v - p^{m-1} t_1)}{t_1(p^m - q^m)} \left[\mathbf{r}_l(t_1, \ldots, t_l) - \mathbf{r}_l((q/p)t_1, \ldots, t_l)\right] \tag{4}$$

gives parametrization of the part of variety $\mathcal{V}_{l+1}$, *on which there exists* $\mathrm{Ch}_{p:q}^{(m-1)}(t_1)$, *simple root* v *and other chains of roots are the same as on the initial variety* $\mathcal{V}_l$.

From the geometrical point of view Theorem 2 means that part of variety $\mathcal{V}_{l+1}$ is formed as ruled hypersurface by the secant lines, which cross its directrix $\mathcal{V}_l$ at two points defined by such values of parameters t_1^1 and t_1^2 such that $t_1^1 : t_1^2 = q : p$. If polynomial $f_n(x)$ has on the variety $\mathcal{V}_{l+1}$ pairs of complex-conjugate roots it is necessary to make continuation of obtained parametrization (4).

Let start from partition $\left[n^1\right]$ which corresponds to variety $\mathcal{V}_1$ with the only chain $\mathrm{Ch}_{p:q}^{(n)}(t_1)$ of roots on it. One can apply transformation (4) of the Theorem 2 in succession and finally can obtain parametrization of variety $\mathcal{V}_{n-1}$ of the maximal dimension $\dim \mathcal{V}_{n-1} = n - 1$. There exists only one chain of roots of the length 2 on it and other roots are simple.

Let define the following three operations, which make it possible to obtain parametrization of each variety $\mathcal{V}_l$ of dimensions from 2 to $n - 1$.

ASCENT: allows to pass from variety $\mathcal{V}_i$ to the part of another variety $\mathcal{V}_{i+1}$ with dimension one greater.

CONTINUATION: allows to get the parametrization of the entier variety $\mathcal{V}_{i+1}$ obtained on the previous step.

DESCENT: allows to pass from variety $\mathcal{V}_j$, on which there exist two chains of roots with equal length, say k, to variety $\mathcal{V}_{j-1}$, on which there exists a chain of roots with length $2k$.

One can combine successively mentioned above operations to obtain parametric representation of each variety $\mathcal{V}_i$ from the resonance set (2).

Statement 1. *Resonance set* $\mathcal{R}_{p:q}(f_n)$ *of real polynomial* $f_n(x)$ *for a certain value of commensurability coefficient* $p : q$ *allows polynomial parametrization of each its variety* $\mathcal{V}_l \subset \mathcal{R}_{p:q}(f_n)$.

3. Resonance set of cubic polynomial

Consider real cubic polynomial

$$f_3 = x^3 + a_1 x^2 + a_2 x + a_3. \tag{5}$$

It has two generalized subdiscriminants

$$\mathrm{GD}^{(1)}_{p:q}(f_3) = pqa_1^2a_2 + (p^2+pq+q^2)a_1a_3 - (p+q)^2a_2^2,$$

$$\mathrm{GD}^{(0)}_{p:q}(f_3) = (pq(p+q))^2\,a_1^3a_3 - q^3p^3a_1^2a_2^2 - pq\left(p^2+pq+q^2\right)\times$$
$$\times\left(p^2+4\,pq+q^2\right)a_1a_2a_3 + (pq(p+q))^2\,a_2^3 + \left(p^2+pq+q^2\right)^3a_3^2.$$

Resonance set $\mathcal{R}_{p:q}(f_3)$ shown in Figure 1 consists of two varieties

$$\mathcal{V}_1 : \left\{a_1 = -(p^2+pq+q^2)t_1,\ a_2 = pq(p^2+pq+q^2)t_1^2,\ t_3 = -(pqt_1)^3\right\},$$

$$\mathcal{V}_2 : \left\{a_1 = -(p+q)t_1 - t_2,\ a_2 = pqt_1^2 + (p+q)t_1t_2,\ a_3 = -pqt_1^2t_2\right\},$$

which corresponds to partitions $\left[3^1\right]$ and $\left[1^12^1\right]$ respectively.

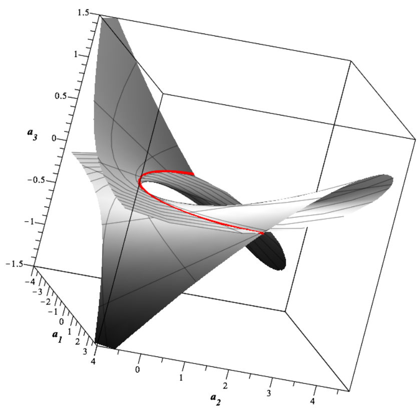

Figure 1. Resonance set $\mathcal{R}_{7:1}(f_3)$.

References

[1] A.B. Batkhin, *Segregation of stability domain of the Hamilton nonlinear systems* // Automation and Remote Control (2013), Vol. 74, No. 8, pp. 1269–1283.

[2] A.B. Batkhin, *Parametrization of the discriminant set of a real polynomial.* Preprints of KIAM. 2015. No. 76. 36 p. (in Russian) URL: `http://library.keldysh.ru/preprints.asp?id=2015-76`

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