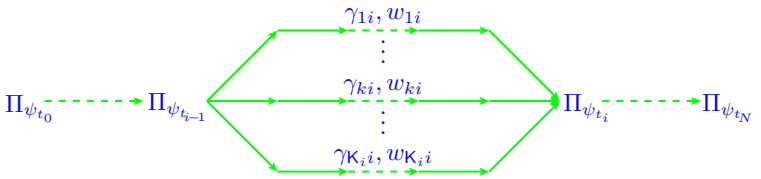
Emergence of quantum structures in permutation dynamics

Vladimir V. Kornyak

Discrete model of quantum evolution. The trajectory of a quantum system is a sequence of unitary evolutions interspersed with observations. We consider a model of quantum dynamics represented by the following scheme



(1)

Here $t_0, \ldots, t_N \in \mathbb{Z}$ are *times of observations* $\Pi_{\psi_{t_0}}, \ldots, \Pi_{\psi_{t_N}}$ represented by *projectors*, namely, $\Pi_{\psi_{t_i}} = |\psi_{t_i}\rangle\langle\psi_{t_i}|$ is the projector that fixes $\psi_{t_i} \in \mathcal{H}$ (a Hilbert space) as the result of observation at the time t_i. The entities γ_{ki} are all possible products of $\Delta t_i = t_i - t_{i-1}$ elements of a finite *gauge group* $\mathsf{G} = \{\mathsf{g}_1, \ldots, \mathsf{g}_\mathsf{M}\}$ with unitary representation U in $\mathcal{H}$. $w_{ki} \geq 0$ is the *weight* of γ_{ki}, $\sum_{k=1}^{K_i} w_{ki} = 1$. The parallel transports (gauge connections) γ_{ki} describe different ways of identification of the indistinguishable objects in the transition between the instants t_{i-1} and t_i.

Single unitary evolution as the most likely gauge connection. Standard quantum mechanics implies a single unitary evolution between measurements, i.e., for some $a \in \mathsf{G}$ the product $a^{\Delta t_i}$ has weight 1, and all other products have zero weights. The unitary evolution can be written as $U = \mathrm{U}\left(a^{\Delta t_i}\right)$, or, introducing the *Hamiltonian* $H = \mathrm{i} \ln \mathrm{U}(a)$, as $U = \mathrm{e}^{-\mathrm{i}H\Delta t_i}$. However, the fact that the gauge fields in physics are determined by the least action principle leads to the idea to study the case of general weights suggesting that a unique unitary evolution should occur as a dominant element in the set of all possible evolutions.

Selection of most likely trajectories and the principle of least action. The main problem in the study of the evolution is the search of the most probable trajectories. The *one-step transition probability* is given by the formula

$$\mathbf{P}_{\psi_{t_{i-1}} \to \psi_{t_i}} = \sum_{k=1}^{\mathsf{K}_i} w_{ki} \left\langle \varphi_{ki} \right| \Pi_{\psi_{t_i}} \left| \varphi_{ki} \right\rangle, \quad \text{where} \quad \varphi_{ki} = \mathrm{U}(\gamma_{ki})\, \psi_{t_{i-1}}.$$

The probability of the whole trajectory can be calculated as the product

$$\mathbf{P}_{\psi_{t_0}\to\cdots\to\psi_{t_N}} = \prod_{i=1}^{N} \mathbf{P}_{\psi_{t_{i-1}}\to\psi_{t_i}}. \tag{2}$$

It is convenient to introduce the *one-step entropy* $\mathbf{S}_{\psi_{t_{i-1}}\to\psi_{t_i}} = \log \mathbf{P}_{\psi_{t_{i-1}}\to\psi_{t_i}}$ and replace the product of probabilities (2) by the *entropy of trajectory*

$$\mathbf{S}_{\psi_{t_0}\to\cdots\to\psi_{t_N}} = \sum_{i=1}^{N} \mathbf{S}_{\psi_{t_{i-1}}\to\psi_{t_i}}.$$

This is a discrete counterpart of the continuous *action* $\mathcal{S} = \int \mathcal{L} dt$.

Evolution of gauge connections. Let $\{w_m^t\}$, $1 \le m \le \mathsf{M}$ be a probabilistic distribution (i.e., $w_m^t \ge 0$ and $w_1^t + \ldots + w_{\mathsf{M}}^t = 1$) on the gauge group G at the time t. Consider a group algebra element of the form $A = w_1^1 \mathsf{g}_1 + \ldots + w_{\mathsf{M}}^1 \mathsf{g}_{\mathsf{M}}$. The time evolution of the statistics of parallel transports is described by the formula

$$w_1^t \mathsf{g}_1 + \ldots + w_{\mathsf{M}}^t \mathsf{g}_{\mathsf{M}} = A^t.$$

The uniform distribution, obviously, defines an idempotent in the group algebra: $B = \frac{1}{\mathsf{M}} (\mathsf{g}_1 + \ldots + \mathsf{g}_{\mathsf{M}}) \Longrightarrow B^t = B$. A^t tends at large time $(t \gg \operatorname{Exp} \mathsf{G})$ to the uniform distribution on the group generated by the elements with nonzero initial probabilities. For small time the evolution of statistics resembles the statistics evolution of a random walk. The behavior of the model — as is typical for quantum mechanics — depends on the choice of time intervals between measurements Δt_i.

Natural and standard representations of symmetric group. The most important and universal example of a finite group is the symmetric group S_{N}. We use here the N-dimensional *natural* and $(\mathsf{N}-1)$-dimensional irreducible *standard* representations of S_{N} to study the above model. For our purposes it is sufficient to consider the constructive version of natural representation that is obtained by replacing the complex space $\mathbb{C}^{\mathsf{N}}$ by the module $\mathbb{N}^{\mathsf{N}}$, where $\mathbb{N} = \{0, 1, \ldots\}$. Then, the constructive version of standard representation is obtained by the projection. As a test case, we consider the *quantum Zeno effect* for these representations within the framework of our approach.

Energy of permutation. Planck's formula $E = h\nu$ associates the energy with the periods of underlying microscopic processes. By analogy, we define the *"energy"* of a permutation p as its frequency: $\varepsilon_p = \dfrac{1}{\operatorname{ord} p}$. Of course, this definition is only approximate because the unitary operator associated with the permutation contains many frequencies: they are inverses of the lengths $\ell_1, \ldots, \ell_K$ of disjoint cycles that constitute the permutation and the period of the permutation is equal to $\operatorname{ord} p = \operatorname{lcm}(\ell_1, \ldots, \ell_K)$. However, the definition is useful for classifying the elements of permutation groups from a "physical point of view". Moreover, the permutations of the dominant conjugacy classes usually contain a small number of different frequencies and the definition becomes "almost exact".

Example. Computations with S_{50}. FIGURE 1 shows the "energy spectrum" for the dominant classes of the group S_{50} in the range of periods $[1..300]$. The numerical characteristics of the group are: $|\mathsf{S}_{50}| = 50! \approx 3.04 \times 10^{64}$; the number of conjugacy classes is equal to 204226; the maximum period is equal to 180180. The size of the largest class is approximately equal to 6.2×10^{62}, and its period is 49. Data for

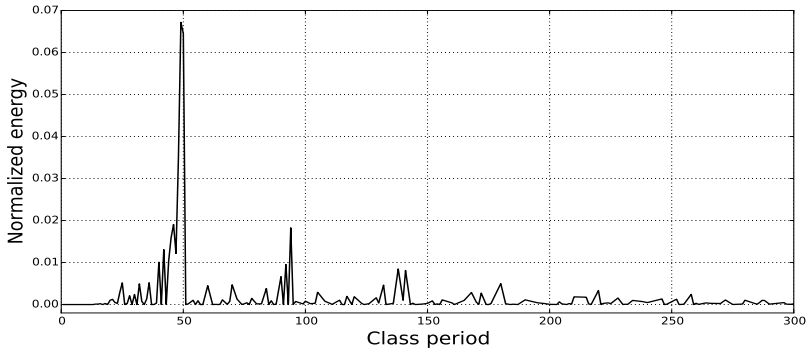


FIGURE 1. Energy spectrum of S_{50}. Exact computation.

the graph were obtained by exact calculations.

Monte Carlo approach. The study of the model discussed here by analytical methods seems to be problematic. A reasonable approach is to use Monte Carlo methods, which in their spirit are perfectly adequate to the *"irreducible quantum randomness"* (John von Neumann[1]). FIGURE 2 reproduces data obtained from the

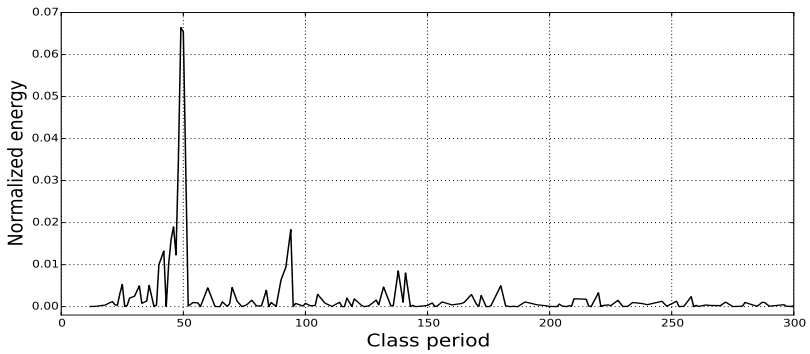


FIGURE 2. Energy spectrum of S_{50}. Randomly generated 10^6 permutations.

[1]Both the idea of the irreducibility of quantum randomness and Monte Carlo methods were actively promoted by John von Neumann.

Monte Carlo simulation. In this task — which took about 10 sec on 3.3 GHz Intel Core i3-2120 CPU — we have generated 10^6 elements of the group S_{50}. Comparing the figures, we see that the Monte Carlo simulation reproduces the general features of the distribution rather satisfactory taking into account that the ratio of the size of the generated sample to the size of the group is only about 10^{-58}.

References

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[2] Kornyak V.V. *Discrete dynamical models: combinatorics, statistics and continuum approximations.* Mathematical Modelling and Geometry, 2015, **3**, No 1, pp. 1–24; available at `http://arxiv.org/abs/1501.07356`

Vladimir V. Kornyak
Laboratory of Information Technologies
Joint Institute for Nuclear Research
Dubna, Russia, 141980
e-mail: `kornyak@jinr.ru`