On the separability of 2-qubit $X-$states

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Abstract. It is shown, that there exists 4-parametric family of separable mixed $X-$states of 2-qubits with an arbitrary spectrum of the density matrix.

Introduction

Consider the Hilbert space $\mathcal{H}$ of binary quantum system, represented by the tensor product of 2-dimensional Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and known under the name of 2-qubit system. The density matrix ϱ , describing mixed states of system $\mathcal{H}$, is *separable* if it allows the convex decomposition:

$$\varrho = \sum_k \omega_k \varrho_1^k \otimes \varrho_2^k \,, \qquad \sum_k \omega_k = 1, \quad \omega_k > 0 \,, \tag{1}$$

where ϱ_1^k and ϱ_2^k represent the density matrices, acting on the corresponding multiplier of $\mathcal{H}$. Otherwise it is *entangled* [1] .

Are there mixed states which are separable for an arbitrary spectrum of ϱ *?* The present note aims to prove, that the answer to this question is affirmative for the wide class of density matrices, describing the so-called $X-$ states (see [2] and the modern review [3] for details). Furthermore, it will be shown, that among the elements of generic 7-dimensional space of $X-$ states with given spectrum, one can point a special 4-parametric family of separable density matrices.

1. Eigenvalue decomposition for $X-$ states

To formulate the statement precisely, consider the density matrices of the form:

$$\varrho_X := \begin{pmatrix} \varrho_{11} & 0 & 0 & \varrho_{14} \\ 0 & \varrho_{22} & \varrho_{23} & 0 \\ 0 & \varrho_{32} & \varrho_{33} & 0 \\ \varrho_{41} & 0 & 0 & \varrho_{44} \end{pmatrix} . \tag{2}$$

In (2) the diagonal entries are real numbers, while elements of the minor diagonal are pairwise complex conjugated, $\varrho_{14} = \overline{\varrho}_{14}$ and $\varrho_{23} = \overline{\varrho}_{32}$. Since non zero elements in the density matrix (2) are distributed in a shape that reminds the Latin letter "X", the corresponding states are named as the $X-$states.

It can be proved, that the matrix (2) is unitary equivalent to the diagonal matrix

$$\varrho_X = KWP\,\mathrm{diag}\,(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\ PW^{+}K^{+}\,, \tag{3}$$

where the spectrum $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ forms the partially ordered simplex, $\underline{\Delta}_3$, i.e.,

$$\underline{\Delta}_3: \quad \sum_{i=1}^{4}\lambda_i = 1 \qquad 0\leq\lambda_2\leq\lambda_1\leq 1, \quad 0\leq\lambda_4\leq\lambda_3\leq 1, \tag{4}$$

and

$$W = \left(\begin{array}{c|c} e^{i\frac{\phi_1}{2}\sigma_2} & 0 \\ \hline 0 & e^{i\frac{\phi_2}{2}\sigma_2} \end{array}\right), \ P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{5}$$

The matrix K in (3) is the element from the subgroup $SU(2)\times SU(2)\subset SU(4)$ of the form $K=\exp{(i\frac{u}{2}\sigma_3)}\times\exp{(i\frac{v}{2}\sigma_3)}$. Here σ_2 and σ_3 are the standard 2×2 Pauli matrices.

2. Applying the Peres-Horodecki separability criterion

Having representation (3), one can analyse the separability as a function of density matrices eigenvalues $\{\lambda\}$. According to the Peres-Horodecki criterion [4], which is a necessary and sufficient condition of separability for 2×2 and 2×3 dimensional systems, a state ϱ is separable iff its partial transposition, i.e., $\varrho^{T_2} = I\otimes T\varrho$, is the semi-positive as well.[1] Applying the Peres-Horodecki criterion to the density matrix (3), one can verify that the $X-$state density matrix is separable iff:

$$(\lambda_1-\lambda_2)^2\cos^2\phi_1 + (\lambda_3-\lambda_4)^2\sin^2\phi_2 \leq (\lambda_1+\lambda_2)^2, \tag{6}$$

$$(\lambda_3-\lambda_4)^2\cos^2\phi_2 + (\lambda_1-\lambda_2)^2\sin^2\phi_1 \leq (\lambda_3+\lambda_4)^2. \tag{7}$$

Introducing new variables

$$x = (\lambda_1-\lambda_2)^2\cos^2\phi_1, \quad y = (\lambda_3-\lambda_4)^2\cos^2\phi_2, \tag{8}$$

the inequalities (6) and (7) linearize

$$\begin{cases} x-y\leq a, & 0\leq x\leq c, \\ y-x\leq b, & 0\leq y\leq d, \end{cases} \tag{9}$$

with parameters (a,b,c,d), obeying the inequalities:

$$a+b\geq 0, \quad a+d\geq 0, \quad b+c\geq 0. \tag{10}$$

[1]Here we consider the partial transposition with respect to the ordinary transposition operation, T, in the second subsystem, similarly one can use the alternative action, $\varrho^{T_1} = T\otimes I\varrho$.

The parameters (a, b, c, d) are functions of the density matrix eigenvalues

$$a = (\lambda_1 + \lambda_2)^2 - (\lambda_3 - \lambda_4)^2, \qquad c = (\lambda_1 - \lambda_2)^2,$$
$$b = -(\lambda_1 - \lambda_2)^2 + (\lambda_3 + \lambda_4)^2, \qquad d = (\lambda_3 - \lambda_4)^2\,.$$

Now it is an easy task to be convinced that (9) have solutions for all possible values of parameters from (10). In other words, for eigenvalues from the partially ordered simplex (4) the inequalities (6) and (7) determine non empty domain of definition for angles ϕ_1 and ϕ_2 in (3). Typical domains on the $(x, y)-$plane are depicted on the FIGURE 1.

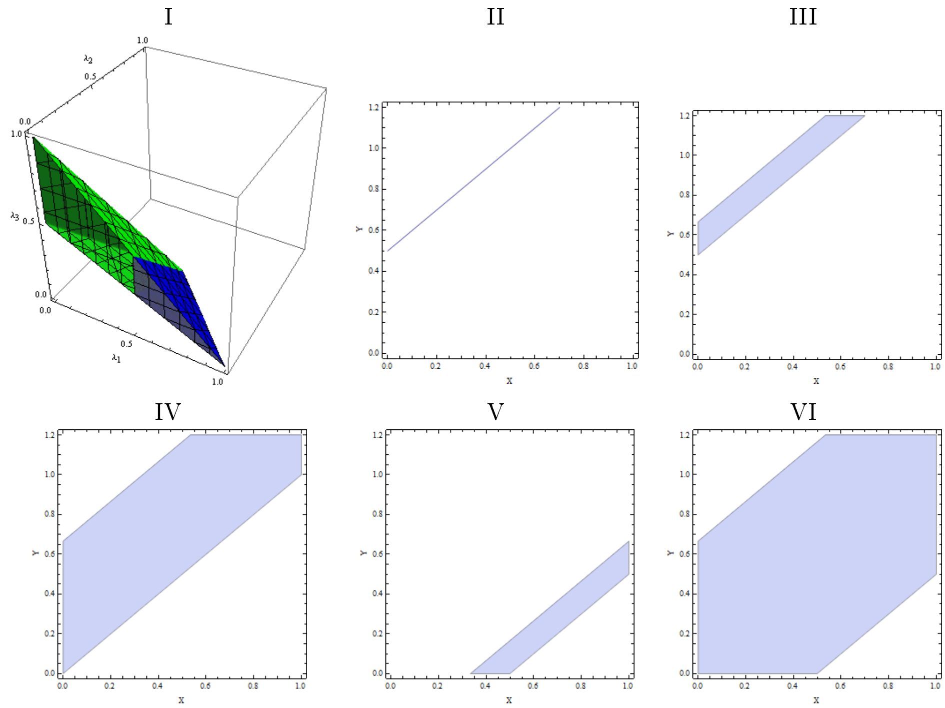

FIGURE 1. Plot (I) - the partially ordered simplex $\underline{\Delta}_3$. Plots (II-VI) - families of solutions to the Eqs. (9) corresponding to the following decomposition of $\underline{\Delta}_3$: Domain (II) : $a < 0$, $b = -a$, $c \geq 0$, $d \geq b$; Domain (III): $a < 0$, $b > -a$, $c \geq 0$, $d \geq -a$; Domain (IV): $a = 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$; Domain (V): $a > 0$, $-a \leq b \leq 0$, $c \geq -b$, $d \geq 0$; Domain (VI): $a > 0$, $b > 0$, $c \geq 0$, $d \geq 0$.

Conclusion

The dependence of separability from the spectrum of 2-qubit density matrix for $X-$states has been discussed. Here it is in order to make few comments on possible generalization of obtained results to an arbitrary 2-qubit states. The Peres -Horodecki separability criterion can be written in the form of polynomial inequalities in the $SU(2)\times SU(2)-$invariant polynomials, (the determinants of correlation and Schlienz -Mahler matrices) [5], [6], treated as unknown variables, and $SU(4)$ Casimir invariants considered as free parameters. [2] Following the analysis, given in the present note, one can conjecture that these inequalities constraint only these two $SU(2) \times SU(2)-$invariant polynomial measures of entanglement.

References

[1] R.F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. **A 40**, 4277-4281, (1989),

[2] Ting Yu and J.H. Eberly, Evolution from entanglement to decoherence of bipartite mixed "X" states. Quantum Information and Computation, **7**, 459-468, (2007).

[3] P. Mendonca, M. Marchiolli and D. Galetti, Entanglement universality of two-qubit $X-$states. Annals of Physics. **351**, 79-103, (2014).

[4] A.Peres, Separability criterion for density matrices, Phys. Rev. Lett. **77** 1413-1415 (1996).

[5] V. Gerdt, A. Khvedelidze and Yu. Palii, On the ring of local polynomial invariants for a pair of entangled qubits, Zapiski POMI **373**, 104-123, (2009).

[6] V. Gerdt, A. Khvedelidze and Yu. Palii, Constraints on SU(2)×SU(2) invariant polynomials for entangled qubit pair, Phys. of Atomic Nucleii, Vol. 74, 893-900, (2011).

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[2] It is assumed, that the Hermicity and semi-positivity of the density matrix is taken into account.