Dzhanibekov's flipping nut and Feynman's wobbling plate

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Abstract. We demonstrate that explicit highly efficient formulas of motion of a freely rotating rigid body, as observed from an inertial frame, might be obtained (only) after exploring (all) the "symmetries" of its motion. We shall disclose the marvel of Galois' construction (which had evaded Poinsot) behind the final step towards a conclusive analytic solution!

Introduction

Vladimir Arnold discussed "the motion of a rigid body, in the absence of outside forces". He wrote, in [5, p. 146],[1] that

"The second revolution will be exactly like the first; if $\alpha = 2\pi p/q$, the motion is completely periodic; if the angle is not commensurable with 2π, the body will never return to its initial state."

Arnold is referring to Poinsot's construction, which (geometrically) describes the "trajectory" of the "tip" of the angular velocity of a freely moving rigid body. The said angle α is the angle of rotation of the body about the (fixed) angular momentum as the angular velocity (pseudo) vector returns to its initial state in body's (rotating) frame. Aside from implicitly presuming (unknown) integers, neither in the Russian nor in the English edition, did Arnold tell us what p and q were. He told us nothing more on calculating α. Lev Landau, on the other hand, did not undermine the calculation of that angle, nor did he divert our attention to "philosophical" statements concerning eternity, but honestly (twice) admitted the "complexity" in [10, p. 119],[2] and provided further reference. One might go on to trace this issue back to the tragedy when Poinsot joined those who failed to appreciate the exceptional significance of Galois' contributions (in uniting

[1] The next (translated) statement appears on page 130 of the third Russian edition (1989).
[2] The confession is (twice) made on page 155 of the fourth Russian edition (1988).

algebra with analysis), and (consequently!)[3] failed to arrive at an explicit formula for a (body) rotation matrix as a function of time. And, in particular, an explicit formula for calculating the rotation angle (which Arnold had denoted by α) remained out of reach!

Key formulas and calculations

Let A, B and C denote the principal moments of inertia, whereas let w and m denote the angular velocity and the (fixed) angular momentum, respectively. We shall exploit the same letters to denote the corresponding magnitudes, so that consistently with this notation we might express that square of the angular speed as $w^2 = w{\cdot}w$ and the square of the (constant) magnitude of the angular momentum as $m^2 = m \cdot m$, where the dot $(\cdot)$ denotes the scalar product. Let h denote the (constant) scalar product $m{\cdot}w$, that is twice the kinetic energy. We might initially assume that the moments of inertia A, B and C are pairwise distinct, and we might impose the additional assumption that $A_2B_2C_2 \neq 0$, where

$$A_2 = Ah - m^2,\ B_2 = Bh - m^2,\ C_2 = Ch - m^2.$$

There are two general cases here, namely, the case $B_2 < 0$ for which we impose the ordering $A < B < C$, and the case $B_2 > 0$ for which we impose the ordering $A > B > C$. The projection of the angular velocity w onto body's rotating frame is (doubly) periodic, with quarter period

$$T = \frac{\sqrt{ABC}\,\pi}{2\,M\left(\sqrt{(B-C)\,A_2},\sqrt{(A-C)\,B_2}\right)},$$

where $M(x,y)$ is arithmetic-geometric mean of x and y. Evidently, the scalar function w^2 is, as well, (doubly) periodic (in any reference frame). Now and here, on this PCA 2016, April 18-23, annual conference in St. Petersburg, Russia, we shall present a simple and powerful formula (which we must attribute to Évariste Galois!)[4] for (highly efficiently) calculating the afore-discussed angle α as

$$\frac{\alpha}{4} = \frac{1}{m}\left(h\,T + \int_0^T \frac{A_2B_2C_2\,dt}{ABC\,(m^2w^2 - h^2)}\right) =$$

$$= \frac{T}{m}\left(h + \frac{B_2\,C_2}{m^2(B-C)}\,N\left(\frac{(B-C)\,A_2}{(A-C)\,B_2},0,\frac{m^2(B-C)}{m^2(B-C)+C\,B_2},\frac{m^2(B-C)}{C\,B_2}\right)\right),^{5}$$

[3]Poinsot, having deprived himself from realizing the crucial relevance of Galois' (deeply constructive) ideas for relating the angular velocity to a (corresponding) rotation matrix, was unable to carry out that last necessary step towards the final solution!

[4]Justice will be served if we attribute that said last step (which could not be accomplished by Poinsot) to Galois himself, who would have had undoubtedly carried it out had he been given a chance! Relevant details on Galois' amazing (yet far from fully appreciated) contribution to elliptic functions and modular equations are given in [3].

[5]Observe that the function α is homogeneous of degree 0 whether viewed as a function of the (principal) moments of inertia (for fixed energy and momentum), or as a function of h and m^2

where the function $N(x,\, a, b, c)$ is defined recursively via the relation

$$N(x,\, a, b, c) = N\Big(\sigma(x,1),\, \sigma(x,a,c),\, \sigma(x,b,c),\, \sigma(x,c)\Big),$$

$$\sigma(x,y) := \sigma(x,y,y),\;\; \sigma(x,y,z) := \frac{(\sqrt{x}+y)\,(\sqrt{x}+z)}{2\,(y+z)\,\sqrt{x}}.$$

The value of this recursive function is the limit obtained from successively applying (linear) fractional transformations $L(\cdot, a, b, c)$ either to (successive) first arguments x, thereby generating the sequence $\{L(x,a,b,c) := (b-c)(x-a)/((b-a)(x-c))\}$, or to the (constant) value 1,[6] generating the sequence $\{L(1,a,b,c)\}$. Both sequences converge quadratically to their common point, as shown in [1].

Many limiting cases of the formula for calculating α might (and must) be considered but the first (and foremost) is the critical case with strictly vanishing B_2.[7] This case corresponds to a critical separating solution which is missed by Arnold and many others who (innocently) presumed that the case with $\alpha = \infty$ might be safely ignored. We are now being vividly reminded of this omission after a striking observation, made in 1985 (June 25$^{\text{th}}$) by the Soviet cosmonaut Vladimir Dzhanibekov, of a motion in proximity to a critical separating solution. A video demonstration from an orbiting space station is provided in [7]. That observation had gotten the attention of Terence Tao, who shared his interpretation of the phenomenon publicly on Google+ [9]. Another special case of motion was popularized by Richard Feynman in [8], and was subsequently referred to as the "Feynman's wobbling plate". The declared (by Feynman) spin to wobble ratio (2:1) was corrected by Benjamin Chao in 1989 (after Feynman's death) in [6]:

"A torque free plate wobbles twice as fast as it spins when the wobble angle is slight. The ratio of spin to wobble rates is 1:2 not 2:1!".[8]

Of course, another limiting case of our formula readily applies to an axially symmetric rigid body, rotating about its axis of symmetry, that is, $Ch = m^2 = C^2w^2$ and $C = \lambda\, B = \lambda\, A$, with (constant) $\lambda \in [0,2]$. The spin to wobble ratio is then 1:λ. It is, indeed, 1:2 for a "flat plate",[9] and 1:0 for a "rod" (with vanishing C), as was also rightfully noted by Chao. We might further formalize the definition of the spin to wobble ratio as $4hT/(m\alpha)$, in order to extend it to non-symmetric rigid bodies where we observe that the said ratio is

(for fixed moments of inertia). This property might be expressed via the relation

$$\alpha\left(\lambda A, \lambda B, \lambda C, \mu h, \mu m^2\right) = \alpha\left(A, B, C, h, m^2\right),$$

where λ and μ might (unnecessarily!) be restricted to be positive.

[6]The sequence of first arguments $\{x\}$ converges quadratically to 1.

[7]With B, as we must reemphasize, is the strictly(!) middle moment of inertia.

[8]Having investigated the so-called Chandler wobble phenomenon, Chao knew the correct ratio before he came across Feynman's error.

[9]The ratio is (exactly) 1:2 (only) when "the wobble angle" is strictly zero, that is, when the plate "does not wobble", not unlike the case where the "small-angle approximation of the period of the simple pendulum" turns out being exact (and unambiguously defined!) for calculating the period of the "resting" pendulum, as explained in [2].

strictly less than 1 if $B_2 < 0$. This ratio nears 1 for that flipping nut (which Dzhanibekov observed). The limit ratio at 1 is actually attained at a critical separating solution (as B_2 strictly vanishes)[10]. With this formally extended definition of the ratio, we are ready to assert that it (as ought be) is 1 for a "totally symmetric" rigid body (with $A = B = C$).

Conclusion

Two distinct classes of motion of a freely moving rigid body are separated by critical solutions,[11] which we shall explicitly demonstrate at this conference. The belief that uniqueness of solution (corresponding to a unique trajectory of motion) ought be determined by initial motion conditions will be scrutinized! Most significantly, the presented formula enables not merely calculating the angle α but might readily be adapted for (highly efficiently) constructing an orthogonal transition matrix from body's (rotating) frame to observer's (fixed) frame,[12] with the time domain (necessarily) compactified by adjoining the point at (complex) infinity, as we were incessantly reminded by Dmitry Abrarov (as clarified in [2]).

References

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[10] Although, then $\alpha = \infty$ as was already said.

[11] As was the case with the simple pendulum which oscillatory and rotary motion regimens are separated by Abrarov's critical solution, as discussed in [2].

[12] The quarter period T in the first line of the formula (for α) might be replaced with the "current" time t for determining "intermediate" rotation angle values. Highly efficient calculation of a corresponding (incomplete) integral relies on combining the expression in the second line (for calculating complete integral) with "dividing" elliptic arcs, as done in [4].

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