On the Generating Function of Discrete Chebyshev Polynomials

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Abstract. We give a closed form for the generating function of the discrete Chebyshev polynomials. The closed form consists of the MacWilliams transform of Jacobi polynomials together with a binomial multiplicative factor. It turns out that the desired closed form is a solution to a special case of Heun differential equation.

Introduction

The discrete Chebyshev polynomials belong to the rich family of *orthogonal polynomials* (see [7] for a general treatise on the orthogonal polynomials and [2] for a previous work of the authors).

The sum and the scalar product in $\mathcal{P}_N$ are defined pointwise, and the inner product is defined as

$$\langle p, q\rangle_w = \sum_{l=0}^{N} w_l p(l) q(l). \tag{0.1}$$

The *Krawtchouk polynomials* (see [5]) are orthogonal with respect to weight function $w_l = \binom{N}{l}$ and the discrete Chebyshev polynomials with respect to weight function $w_l = 1$ for each l.

As (orthogonal) polynomials with ascending degree, the discrete Chebyshev polynomials form a basis of $\mathcal{P}_N$, and hence any polynomial p of degree at most N can be uniquely represented as

$$p = d_0 D_0^{(N)} + d_1 D_1^{(N)} + \ldots + d_N D_N^{(N)}, \tag{0.2}$$

where $d_l \in \mathbb{C}$. Since the discrete Chebyshev polynomials are orthogonal with respect to constant weight function, they have the following property important in the approximation theory: With respect to norm $||p-q||^2 = \sum_{l=0}^{N}(p(l)-q(l))^2$, the best approximation of p in $\mathcal{P}_M$ can be found by simply taking $M+1$ first summands of (0.2) (see [4], for instance).

1. Preliminaries

1.1. The Discrete Chebyshev Polynomials

There are various ways to construct polynomials orthogonal with respect to inner product (0.1) with weight function $w_l = 1$ so that $\deg(D_k^{(N)}) = k$.

We choose a construction analogous to that of *Legendre polynomials* [7]. We first define the difference operator Δ by $\Delta f(x) = f(x+1) - f(x)$, the binomial coefficient by $\binom{x}{k} = \frac{1}{k!}x(x-1)\dots(x-k+1)$, and finally

$$D_k^{(N)}(x) = (-1)^k \Delta^k \left(\binom{x}{k} \binom{x-N-1}{k} \right). \tag{1.1}$$

The following recurrence relation is quite easy to verify:

$$k^2 D_k = (2k-1) D_1 D_{k-1} - (N+k)(N-k+2) D_{k-2}, \tag{1.2}$$

$D_0 = 1$, $D_1 = N - 2x$ (see [3]). The recurrence (1.2) also extends the definition of D_k to cases $k > N$.

The method of using *generating functions* is among the cornerstones of various areas of mathematics, and does not need any further introduction. On the other hand, the quest for the generating function of the discrete Chebyshev polynomials seems to be a more complicated task. In what follows, we give a closed form for the generating function

$$\sum_{k=0}^{\infty} D_k^{(N)}(x) t^k. \tag{1.3}$$

1.2. A Differential Equation for Jacobi Polynomials

For a nonnegative integer n, the *Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$ is, up to the constant factor, the unique entire rational solution to the *differential equation (for Jacobi polynomials)*

$$(1-x^2)y'' + \big(\beta - \alpha - (\alpha+\beta+2)x\big)y' + n(n+\alpha+\beta+1)y = 0 \tag{1.4}$$

(see [1]).

In this article, we are interested in Jacobi polynomials with parameters $\alpha = 0$, $\beta = -(N+1)$, where $N > 0$ is a fixed integer. We also substitute x for n and t for x in equation (1.4), and denote $J_x^{(N+1)}(t) = P_x^{(0,-N-1)}(t)$. We usually omit superscript $N+1$ and denote $J_x(t) = J_x^{(N+1)}(t)$. Then $J_x(t)$ satisfies differential equation

$$(1-t^2)J_x''(t) - (N+1-(N-1)t)J_x'(t) + x(x-N)J_x(t) = 0. \tag{1.5}$$

Recall that in this context, x is a fixed nonnegative integer. Polynomial $J_x(t)$ can be expressed as

$$J_x(t) = \frac{1}{2^x} \sum_{k=0}^{x} \binom{x}{k} \binom{x-N-1}{k} (t-1)^k (t+1)^{x-k} \tag{1.6}$$

(see [1]). Since equation (1.5) is clearly invariant under substitution $x \leftarrow N - x$, we have symmetry

$$J_{N-x}(t) = J_x(t) \tag{1.7}$$

1.3. MacWilliams Transform

The *MacWilliams transform* of order x for a polynomial P is defined as

$$\widehat{P}_x(t) = (1+t)^x P(\frac{1-t}{1+t}). \tag{1.8}$$

As definition (1.8) shows, MacWilliams transform is a special case of Möbius transformation together with factor $(1+t)^x$.

In what follows, $\widehat{J}_x(t)$ stands for the MacWilliams transform of J_x of order x. It is then straightforward to uncover a representation for $\widehat{J}_x(t)$:

$$\widehat{J}_x(t) = \widehat{(J_x)}_x(t) = \sum_{k=0}^{x} (-1)^k \binom{x}{k} \binom{x-N-1}{k} t^k. \tag{1.9}$$

The symmetry (1.7) implies straightforwardly

$$\begin{aligned} \widehat{J}_{N-x}(t) &= \widehat{(J_{N-x})}_{N-x}(t) = (1+t)^{N-x} J_{N-x}(\frac{1-t}{1+t}) \\ &= (1+t)^{N-2x}(1+t)^x J_x(\frac{1-t}{1+t}) = (1+t)^{N-2x}\widehat{J}_x(t). \end{aligned}$$

Equality

$$\widehat{J}_{N-x}(t) = (1+t)^{N-2x}\widehat{J}_x(t) \tag{1.10}$$

thus obtained will be important in understanding the alternative representation of the generating function introduced in Section 4.

2. Heun Equation

A differential equation for the MacWilliams transform of $J_x(t)$ can be found easily. For short, we denote $J(t) = J_x(t)$ and $\widehat{J}(t) = \widehat{J}_x(t)$ in the following lemmata.

Lemma 1. *$\widehat{J}(t)$ satisfies differential equation*

$$t(1+t)\widehat{J}''(t) + (Nt + 1 - 2t(x-1))\widehat{J}'(t) + x(x-N-1)\widehat{J}(t) = 0. \tag{2.1}$$

Lemma 2. *Let $T(t)$ be defined as $T(t) = (1+t)^{N-2x}\widehat{J}(-t^2)$. Then $T(t)$ satisfies differential equation*

$$\begin{aligned} (t^3 - t)T''(t) &+ \big(2t(N-2x) + 3t^2 - 1\big)T'(t) \\ &+ \big(N - 2x - tN(N+2)\big)T(t) = 0 \end{aligned} \tag{2.2}$$

3. The Generating Function

By equality (1.9) function $T(t) = (1+t)^{N-2x}\widehat{\mathcal{J}}_x(-t^2)$ can be represented as

$$T(t) = (1+t)^{N-2x}\sum_{k=0}^{x}\binom{x}{k}\binom{x-N-1}{k}t^{2k}. \tag{3.1}$$

We are now ready to state the main result.

Theorem 1. *Function*

$$T_{N,x}(t) = (1+t)^{N-2x}\widehat{\mathcal{J}}_x(-t^2) \tag{3.2}$$

is the generating function of discrete Chebyshev polynomials, i.e. $\tau_k(x) = D_k(x)$ *for each* $k \geq 0$*.*

4. Concluding Remarks

The main result of this article is the closed form 3.2 for Discrete Chebyshev polynomials. As the Discrete Chebyshev polynomials play a major role in approximation theory, this result is evidently interesting on its own. We present also some alternative forms for the generating function:

Theorem 2. *The generating function* $T_{N,x}(t)$ *can be also represented as*

$$T_{N,x}(t) = (1-t)^{2x-N}\widehat{\mathcal{J}}_{N-x}(-t^2). \tag{4.1}$$

To combine Theorems 1 and 2 into a single presentation is straightforward:

Theorem 3 (The explicit polynomial form for $x \in \{0,1,\ldots,N\}$). *The generating function* $T_{N,x}(t)$ *can be presented as a polynomial in* t *of degree* N*:*

$$T_{N,x}(t) = (1+t\cdot\operatorname{sign}(N-2x))^{|N-2x|}\widehat{\mathcal{J}}^{(N)}_{\min\{x,N-x\}}(-t^2).$$

Remark 1. Theorem 1 implies that

$$\sum_{0\leq l\leq k/2}\binom{N-2x}{k-2l}\binom{x}{l}\binom{x-N-1}{l} = \sum_{l=0}^{k}(-1)^l\binom{k}{l}\binom{N-x}{k-l}\binom{x}{l}. \tag{4.2}$$

A direct combinatorial proof of (4.2) appears challenging, for instance, the techniques of [6] appear powerless in this case. Theorem 2 implies an identity similar to (4.2).

References

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