Hankel polynomials in the interpolation problems

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Abstract. We treat the univariate interpolation problem $\{f(x_j) = y_j\}_{j=1}^N$ for polynomial and rational functions. Developing the approach by C.Jacobi, we represent the interpolants by virtue of the Hankel polynomials generated by the sequences $\{\sum_{j=1}^N x_j^k y_j / W'(x_j)\}_{k\in\mathbb{N}}$ and $\{\sum_{j=1}^N x_j^k / (y_j W'(x_j))\}_{k\in\mathbb{N}}$; here $W(x) = \prod_{j=1}^N (x - x_j)$. The obtained results are applied for the error correction problem, i.e. the problem of reconstructing the polynomial from a redundant set of its values some of which are probably erroneous. The problem of evaluation of the resultant of polynomials $p(x)$ and $q(x)$ from the set of their values is also tackled within the framework of this approach.

Introduction

Compared with the polynomial interpolation problem which is known for numerous practical applications since its very beginning in the 17th century, rational interpolation problem has received its real life application points in the late 20th century. In case of infinite fields, these are Control Theory (recovering of the transfer function from the frequency responses) while for the case of finite fields these are Error Correcting Codes (Berlekamp-Welch agorithm).

We solve the polynomial and the rational interpolation problems $\{f(x_j) = y_j\}_{j=1}^N$ with the aid of the Hankel polynomials [1], i.e. polynomials in x with the following representation in the determinantal form

$$\mathcal{H}_k(x; \{c\}) = \left| \begin{array}{lllll} c_0 & c_1 & c_2 & \dots & c_k \\ c_1 & c_2 & c_3 & \dots & c_{k+1} \\ \vdots & & & \ddots & \vdots \\ c_{k-1} & c_k & c_{k+1} & \dots & c_{2k-1} \\ 1 & x & x^2 & \dots & x^k \end{array} \right| = H_k(\{c\})x^k + h_{k1}x^{k-1} + \dots \quad (1)$$

Here the generators $c_0, c_1, \dots c_{2k-1}$ of the polynomial are the elements of some (finite or infinite) field and $H_k(\{c\}) = \det[c_{i+j-2}]_{i,j=1}^k$ is the Hankel determinant.

1. Rational interpolation

We are looking for the rational interpolant in the form $f(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials, $\deg p \le n, \deg g \le m$ and $N = n + m + 1$. The interpolation problem is not always solvable.

Theorem 1. *Let $\{y_j \ne 0\}_{j=1}^N$, and $W(x) = \prod_{j=1}^N (x - x_j)$. Compute the values*

$$\tau_k = \sum_{j=1}^{N} y_j \frac{x_j^k}{W'(x_j)} \qquad \text{for } k \in \{0, \dots, 2m\} \tag{2}$$

and

$$\widetilde{\tau}_k = \sum_{j=1}^{N} \frac{1}{y_j} \frac{x_j^k}{W'(x_j)} \qquad \text{for } k \in \{0, \dots, 2n-2\}\,, \tag{3}$$

and generate the corresponding Hankel polynomials $\mathcal{H}_m(x; \{\tau\})$ and $\mathcal{H}_n(x; \{\widetilde{\tau}\})$. If $H_n(\{\widetilde{\tau}\}) \ne 0$ and $\{\mathcal{H}_m(x_j; \{\tau\}) \ne 0\}_{j=1}^N$ then there exists a unique solution to rational interpolation problem with $f(x) \equiv p(x)/q(x)$ where $\deg p(x) = n, \deg q(x) \le m = N - n - 1$. It can be expressed as:

$$p(x) = H_{m+1}(\{\tau\})\mathcal{H}_n(x; \{\widetilde{\tau}\})\ ,\ q(x) = H_n(\{\widetilde{\tau}\})\mathcal{H}_m(x; \{\tau\})\,. \tag{4}$$

2. Polynomial interpolation

For $m = 0$, Theorem 1 yields a solution to the polynomial interpolation problem. We apply this result to the error correction in the data set, i.e. for the case where the data set $\{(x_j, y_j)\}_{j=1}^N$ is redundant for the interpolant computation ($n < N-1$) but probably contains erroneous values.

Theorem 2. *Let $E \in \{2, 3, \dots, \lfloor N/2 \rfloor - 1\}$. Let polynomial $p(x)$ be of a degree $n < N - 2E$. Let the data set $\{(x_j, y_j)\}_{j=1}^N$ satisfy the conditions*

(a) $y_j \ne 0$ *for* $j \in \{1, \dots, N\}$,
(b) $y_j = p(x_j)$ *for* $j \in \{1, \dots, N\} \setminus \{e_1, \dots, e_E\}$,
(c) $\widehat{y}_{e_s} = p(x_{e_s}) \ne y_{e_s}$ *and* $\widehat{y}_{e_s} \ne 0$ *for* $s \in \{1, \dots, E\}$.

Then

$$\mathcal{H}_E(x; \{\tau\}) \equiv \frac{\displaystyle\prod_{s=1}^{E} (y_{e_s} - \widehat{y}_{e_s}) \prod_{1 \le s < t \le E} (x_{e_t} - x_{e_s})^2}{\displaystyle\prod_{s=1}^{E} W'(x_{e_s})} \prod_{s=1}^{E} (x - x_{e_s})\,. \tag{5}$$

This result means that the sequence of the Hankel polynomials $\{H_k(x; \{\tau\})\}_{k=1}^{N-1}$ contains at least one polynomial with the zero set coinciding with the set of all the nodes x_j corresponding to corrupted values y_j in the data set initially generated by the polynomial $p(x)$. In terms of Coding Theory, polynomial $\mathcal{H}_E(x; \{\tau\})$ can be referred to as the *error locator polynomial.*

3. Recurrence relation for the Hankel determinants

The problem of computation of the parameter dependent polynomials is not a trivial one. Fortunately, the following result concerning the Hankel polynomials was retrieved by the present authors in the 19th century papers:

Theorem 3 (Jacobi, Joachimsthal). *For any generating sequence $\{c\}$, any three consecutive Hankel polynomials $\mathcal{H}_{k-2}(x), \mathcal{H}_{k-1}(x), \mathcal{H}_k(x)$ are linked by the following identity*

$$H_k^2 \mathcal{H}_{k-2}(x) + (H_k h_{k-1,1} - H_{k-1} h_{k1} - H_k H_{k-1} x)\, \mathcal{H}_{k-1}(x) + H_{k-1}^2 \mathcal{H}_k(x) \equiv 0\,. \quad (6)$$

If $H_{k-1} \neq 0$, formula (6) permits one to organize the computation of $\mathcal{H}_k(x)$ recursively in k. We also suggest a result to deal with an exceptional case when $H_{k-1} = 0$. These results can be applied for an effective computation of the whole family of rational interpolants for all the possible combinations for the orders of numerator and denominator satisfying the restriction $\deg p(x) + \deg q(x) = N - 1$.

4. Resultant interpolation

It turns out that the numerical factors in formulas (4) are related to the resultant $\mathcal{R}(p(x), q(x))$ of the polynomials from the numerator and the denominator of the rational function.

Theorem 4. *Let $\{p(x_j) \neq 0, q(x_j) \neq 0\}_{j=1}^N$ and $\deg p(x) = n$. Let the values (2) and (3) be calculated for $\{y_j = p(x_j)/q(x_j)\}_{j=1}^N$. The following equalities are valid*

$$H_n(\{\widetilde{\tau}\}) = \frac{(-1)^{mn+n(n+1)/2} p_0}{\prod_{j=1}^N p(x_j)} \mathcal{R}(p(x), q(x))\,,$$

$$H_{m+1}(\{\tau\}) = \frac{(-1)^{m(m+1)/2} p_0}{\prod_{j=1}^N q(x_j)} \mathcal{R}(p(x), q(x))\,;$$

here p_0 stands for the leading coefficient of $p(x)$.

This result might be useful for the problems where the canonical representations for polynomials $p(x)$ and $q(x)$ are a priori unknown, like, for instance, the problem of establishing the existence of a common eigenvalue for the given pair of matrices.

Conclusion

The problem of interpolation of the given data set $\{y_j\}_{j=1}^N$ by the set $\{p(x_j)\}_{j=1}^N$ of values of a polynomial of a degree $n \ll N$, if treated in $\mathbb{R}$, is frequently tackled by the least squares method. However the LSM is sensitive to the occurrence of *outliers*, i.e. incidental systematic errors. On the basis of the result of Theorem 2, we hope to formulate criteria for distinguishing systematic errors from the non-systematic ones in the “noisy” data set.

The extension of the results of Sections 1–4 to the multivariate case is also a subject of future investigation.

References

[1] P. Henrici, *Applied and Computational Complex Analysis.* V. 1. Wiley, NY, 1974.

[2] A.Yu. Uteshev, and I. Baravy, *Solution of Interpolation Problems via the Hankel Polynomial Construction*, preprint (2016), available at `http://arxiv.org/abs/1603.08752`.

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