On the ring of local unitary invariants for mixed $X-$states of two qubits

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• **Two-qubit quantum states** • Generally, the state of a two-qubit quantum system is described by the density matrix which has the following general structure

$$\varrho = \frac{1}{4}\left[I_2 \otimes I_2 + \sum_{i=1}^{3} a_i \sigma_i \otimes I_2 + \sum_{i=1}^{3} b_i I_2 \otimes \sigma_i + \sum_{i,j=1}^{3} c_{ij} \sigma_i \otimes \sigma_j \right] . \tag{1}$$

where $\sigma_1, sigma_2, sigma_3$ are the Pauli matrices, and I_2 is the unit 2×2 matrix. The 15 real parameters a_i, b_i and c_{ij} , $i, j = 1, 2, 3$, define the space

$$W := \{ (a_i, b_j, c_{kl}) \in \mathbb{R}^{15} \mid i, j, k, l = 1, 2, 3 \} , \tag{2}$$

and the corresponding $SU(2)\times SU(2)-$invariant polynomials accumulate all relevant information on the quantum two-qubit entanglement.

The 15-dimensional space (2) is subject to the physical constraints coming from the semipositivity condition imposed on the density matrix:

$$\varrho \geq 0 . \tag{3}$$

Explicitly, the semipositivity condition (3) reads as a set of polynomial inequalities in the fifteen variables a_i, b_i and c_{ij} , and thereby determines a semialgebraic variety of (2) (see, e.g., [11] and references therein).

• **The research object** • We study the special 7-dimensional subspace of (2), the space of so-called $X-$states [1]. These states got such name due to the visual similarity of the density matrix, whose non-zero entries lie only on the main and minor (secondary) diagonals, with the Latin letter "X":

$$\varrho_X := \begin{pmatrix} \varrho_{11} & 0 & 0 & \varrho_{14} \\ 0 & \varrho_{22} & \varrho_{23} & 0 \\ 0 & \varrho_{32} & \varrho_{33} & 0 \\ \varrho_{41} & 0 & 0 & \varrho_{44} \end{pmatrix} . \tag{4}$$

In (4) the diagonal entries are real numbers, while elements of the minor diagonal are pairwise complex conjugated, $\varrho_{14} = \overline{\varrho}_{14}$ and $\varrho_{23} = \overline{\varrho}_{32}$. Our interest to this subspace of (2) is due to fact that many well-known states, e.g. the Bell states [2], Werner states [3], isotropic states [4] and maximally entangled mixed states [5, 6] are particular subsets of the $X-$states. Since their introduction in [1], many interesting properties of $X-$states have been established. Particularly, it was shown that for a fixed set of eigenvalues the states of maximal concurrence, negativity or relative entropy of entanglement are the $X-$states. [1]

• **Main results** • Here we pose the question about the algebraic structure of the local unitary polynomial invariants algebra corresponding to the $X-$states. More precisely, the fate of generic $SU(2)\times SU(2)$-invariant polynomial ring of 2-qubits [8]–[11] under the restriction of the total 2-qubit state space to its subspace

$$W_X := \{\, w \in W \mid c_{13} = c_{23} = c_{31} = c_{32} = 0\,, a_i = b_i = 0\,,\ i = 1, 2\,\}$$

will be discussed. Our research is based on the classical invariant theory [13] and its computational aspects [14, 15] based on computer algebra. The quotient structure of the ring obtained as a result of restriction will be determined. Furthermore, we establish an injective homomorphism between this ring and the and the ring $\mathbb{R}[W_X]^{SO(2)\times SO(2)}$ of local unitary invariant polynomials for the 2-qubit $X-$states. In doing so, we show that the latter ring is *freely* generated by five homogeneous invariants of degrees 1,1,1,2,2.

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