Generation and annihilation of apparent singularities

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The problem of reconstruction of coefficients for Fuchsian equations from monodromie data had been studied by Plemelj [1]. The main result was negative. In other words the parameters of equations can not be reconstructed from monodromie data. One of the reasons lays in existence of so-called apparent singularities which correspond to appropriate parameters but for which a monodromie matrix is trivial.

Here we give some examples of recipes how to generate or annihilate apparent singularities in linear differential equations with polynomial coefficients by turning to equations for derivatives and inverse derivatives of solutions. Previously this approach was proposed by A. Ishkanyan and Suominen [2] for numerical needs.

First we give the definition of apparent singularity valid for our case. Suppose we have an equation

$$L(D, z)y(z) = \sum_{k=0}^{n} P_k(z)D^k y(z) = 0 \qquad (1)$$

where $L(D, z)$ is a polynomial in two variables: D – differentation operator and z – multiplication by variable z. The corresponding degrees are n and m respectively with $m \geq n+1$. Zeros z_j of the polynomial $P_n(z)$ and the point $z = \infty$ under additional conditions (which are not discussed here) are regular (fuchsian) singularities of equation (1). In this case equation (1) is a fuchsian equation. If two fuchsian singularities coalesce we arrive to irregular singularity and to confluences equation. The latter equations also can be considered as equations with polynomial coefficients.

Among fuchsian singularities apparent singularities can be distinguished in the vicinity of which general solution $y(z)$ is a holomorphic function. We show in examples that if we have equation without apparent singularities the equation for the derivative of a solution possess apparent singularities.

Example 1. Equations of Heun class [3] can be presented as

$$\sigma(z)w''(z) + \tau(z)w'(z) + \omega(z)w(z) - hw(z) = 0, \qquad (2)$$

where $\sigma(z)$ is a polynomial of third degree or less, $\tau(z)$ is a polynomial of second degree or less and $\omega(z)$ is a polynomial of first degree or less. These polynomials depend on parameter t and accessory parameter h.

We introduce beyond the function $w(z)$ its derivative $u(z)$

$$u(z) = w'(z) \qquad (3)$$

Differentiation of (2) leads to the second order equation for $u(z)$

$$\sigma(z)u''(z) + \left(\tau_1(z) - \sigma(z)\frac{\omega'(z)}{\omega(z) - h}\right)u'(z) +$$
$$(\omega_1(z) - h)u(z) - \left(\tau(z)\frac{\omega'(z)}{\omega(z) - h}\right)u(z) = 0 \qquad (4)$$

with $\tau_1(z) = \tau(z) + \sigma'(z)$, $\omega_1(z) = \omega(z) + \tau'(z)$. Equation (2) beyond singularities characteristic for initial Heun equation has an additional apparent singularity $z = q$ being the single zero of $\omega(z) - h$. It had been shown in publication [?] that equation (4) generates nonlinear integrable Painlevé equations.

Example 2. Next in complexity in comparison with Heun equation is an equation similar to (2) but with $\sigma(z) = \prod_{j=1}^{4}(z - z_j)$ - fourth degree polynomial with increase by unity degrees of other polynomials in (2).Clearly the factor $P_0(z)$ in front of $y(z)$ would be a second degree polynomial depending on two accessory parameters. The equation for $u(z) = y'(z)$ will have two apparent singularities according to the degree of mentioned polynomial.

Example 3. Suppose $P_0(z) = A(z-q)^2$ that corresponds to zero with multiplicity 2. Then we arrive after differentiating twice to a single apparent singularity but slightly different from the introduced above. Namely, characteristic indices at apparent singularity are 1 and 3.

What is the meaning of our presentation of apparent singularity. At ordinary point of a differential equation Cauchy data can be posed and afterwards all other derivatives of a solution can be computed via differential equation. However, in the case of apparent singularity the derivative of order equal or higher than the order of the equation can be arbitrary.

Example 4. A particular Fuchsian third order equation with singularities located at the points $z_1 = 0$, $z_2 = 1$, $z_3 = t$ can be considered

$$\begin{aligned} z^2(z-1)(z-t)y'''(z) + [(3-\alpha-\beta)z(z-1)(z-t) - \\ \theta_3 z^2(z-1) - \theta_2 z^2(z-t)]y''(z) + \\ (\alpha-1)(\beta-1)(z-1)(z-t)y'(z) + (\kappa z + h]y(z) = 0. \end{aligned} \quad (5)$$

The Riemann scheme for this equation

$$\begin{pmatrix} z_1 & z_2 & z_3 & \infty & z \\ 0 & 0 & 0 & a & h \\ \alpha & 1 & 1 & b & \\ \beta & 2+\theta_3 & 2+\theta_2 & c & \end{pmatrix}$$

shows the Frobenius characteristic exponents.

$$a + b + c = -\alpha - \beta - (\theta_2 + \theta_3)$$

$$ab + bc + ac = \alpha\beta + (\theta_2 + \theta_3)$$

$$abc = \kappa$$

It means that at finite singularity $z = 0$ there is one holomorphic solution and two solution which are in general not holomorphic whenever at $z = 1$ and $z = t$ there are two linearly independent holomorphic solutions and one which is not holomorphic. The system for the coefficients a, b, c can be explicitly solved Let

$$\theta_2 + \theta_3 = \gamma$$

Then a can be chosen as any root of equation

$$a^3 + a^2(\alpha + \beta + \gamma) + a(\alpha\beta + \gamma) - \kappa = 0$$

and $b = x_1$ and $c = x_2$ are taken as roots of equation

$$ax^2 + (\alpha + \beta + \gamma + a)x + \kappa = 0$$

Differentiating (5) one obtains a third order equation with additional apparent singularity.

The presented examples allow to formulate the following conjectures.

Conjecture 1. *Any linear differential equation with polynomial coefficients containing apparent singularities can be obtained as an equation for a proper derivative of a solution of a linear differential equation without apparent singularities.*

If we turn to inverse derivatives we arrive to the opposite conjecture.

Conjecture 2. *Any linear differential equation with polynomial coefficients containing apparent singularities can be transformed to equation without apparent singularities by use of inverse derivatives of a solution*

To prove this conjectures is an open question.

References

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[3] S.Yu.Slavyanov, W.Lay, "Special Functions: Unified Theory based on Singularities" Oxford University Press, 2000.

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