Coloring triangulated surfaces

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Abstract. A *coloring* of a singular triangulation is a partition of the set of its open simplices into classes (colors). A set of k colorings of one and the same triangulation is called a *k-layered coloring*. It turned out that $(k+m)$-dimensional manifolds and pseudomanifolds can be coded by k-layered colorings of triangulated m-dimensional manifolds. In particular, n-dimensional (pseudo)manifolds can be coded by $(n-1)$-layered colorings of sets of circles. In the first place, we investigate the coding of 3-pseudomanifolds by colorings of triangulated surfaces. Joint research with E. Fominykh and A. Vesnin.

Singular triangulations (simplicial CW-complexes). Let X be a simplicial complex and let Φ be a collection of affine homeomorphisms between some of the faces of X. Following [Mat03, p. 11], we refer to the pair (X, Φ) as to a *face identification scheme* (a *scheme*). A scheme (X, Φ) is called *inadmissible* if homeomorphisms in Φ induce a non-identity automorphism on a face in X. Otherwise, the scheme is called *admissible*. The *quotient space* $Q(X, \Phi)$ of the scheme (X, Φ) is defined as the space obtained from X by identification of faces via the homeomorphisms in Φ. The quotient space of an admissible scheme is naturally endowed with a structure of CW-complex. A CW-complex is said to be *simplicial* if it can be obtained as a quotient space of an admissible scheme.[1] The cell structure of a simplicial CW-complex is called a *singular triangulation*. A simplicial CW-complex X is said to be a *triangulated n-pseudomanifold* if removing of vertices (0-dimensional cell) from X yields an n-manifold.

Colorings. By a *coloring* of a CW-complex X we mean a map $\chi\colon X \to \mathcal{C}$ (where $\mathcal{C}$ is an arbitrary set, called the set of *colors*) which is constant on each open cell of X. We say that a coloring of a CW-complex is *trivial* if no two distinct open cells have the same color. We say that a coloring $\chi\colon X \to \mathcal{C}$ of a topological space X is *coherent* if for each pair of points x, y of the same color there exist arbitrarily small open neighbourhoods U_x and U_y of x and y with a color-preserving homeomorphism $U_x \to U_y$. The trivial coloring is coherent. We say that a coloring

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[1] Cf. *Δ-complexes* in [Hat02, p. 103], *side-pairing* in [Rat94, p. 435], *pseudocomplexes*, etc.

$\chi\colon S \to \mathcal{C}$ of a simplicial CW-complex S is *regular* if for each of its colors $C \in \mathcal{C}$ there exists an integer d such that $\chi^{-1}(C)$ consists of $d+2$ open cells of dimension d.

Derived complexes. Let S be a simplicial CW-complex. Let $V \subset S$ be the set of vertices of S, let $L_2(V)$ be the link of V in the second barycentric subdivision of S, let $\lambda(V)$ be the underlying topological space of $L_2(V)$ with CW-complex structure forgotten, and let $S^\dagger = \lambda^\dagger(V)$ be $\lambda(V)$ with the structure of CW-complex induced by that of S (that is, by the intersections with the cells of S). Clearly, the CW-complex $S^\dagger$ is simplicial. We say that $S^\dagger$ is the *derived* complex of S. If S is a triangulated n-pseudomanifold, then $S^\dagger$ is an $(n-1)$-manifold. If S is a triangulated n-manifold, then $S^\dagger$ is the union of $(n-1)$-spheres. Observe that a coloring of S induces a coloring of $S^\dagger$. A coherent coloring of S induces a coherent coloring of $S^\dagger$. A trivial coloring of S induces a coherent regular coloring of $S^\dagger$. This determines a map F_n from the set of (isomorphism classes of) triangulated n-pseudomanifolds without boundary to the set of (isomorphism classes of) coherent regular colorings of triangulated $(n-1)$-manifolds without boundary.

Theorem 1.

1. *The map F_3 is injective.*
2. *The map F_3 is not surjective.*

Partial colorings. By a *partial coloring* of a CW-complex X we mean a coloring $\chi\colon X \setminus V(X) \to \mathcal{C}$, where $V(X)$ is the set of vertices of X, such that χ is constant on open cells. Let π_2 be the natural forgetting map from the set of coherent regular colorings of singular triangulations on closed surfaces to the set of coherent regular *partial* colorings of singular triangulations on closed surfaces (the map π_2 'forgets' the colors of vertices).

Theorem 2.

1. *The map π_2 is injective.*
2. *The map π_2 is not surjective.*

References

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