Algorithmically generated implicit difference schemes for KdV equation and their numerical behaviour

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• **Introduction** • In this paper we consider the Korteveg-de Vries (KdV) equation in the following form (cf. [1], Eq. 1.18)

$$\partial_t u + \alpha u\, \partial_x u + \beta\, \partial_x^3 u = 0 \tag{1}$$

where $u := u(x,t)$ and $\alpha, \beta \in \mathbb{R}$. Numerical solving of Eq. (1) by the finite difference method was intensively studied in the literature (see book [1] and its bibliography). In so doing, a number of explicit and implicit difference schemes were derived and used for numerical construction of various solutions to (1).

In our talk we consider the Cartesian grid with spacings $\tau := t_{n+1} - t_n$ and $h := x_{j+1} - x_j$ and present two new implicit schemes for Eq. (1) with $\mathcal{O}(\tau^2, h^2)$ and $\mathcal{O}(\tau^2, h^2)$ approximations. The schemes were generated by our algorithmic approach [2] which is based on combination of the methods of finite volumes, numerical integration and difference elimination by means of Gröbner bases. Then we compare, on the exact soliton solution, numerical behavior of these schemes with that of some *classical* schemes of same order of approximation used in the literature and show that our schemes provide substantially better numerical accuracy.

• **Classical schemes with $\mathcal{O}(\tau^2, h^2)$ approximation** • The explicit scheme [1], Eq.1.80

$$u_i^{n+1} = u_i^{n-1} - \frac{\alpha\tau}{h} u_i^n \left(u_{i+1}^n - u_{i-1}^n\right) - \frac{\beta\tau}{h^3}\left(u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n\right). \tag{2}$$

where the standard notation $u_j^n := u(t_n, x_j)$ for the grid function is used. This scheme is stable for

$$\tau \le \frac{2h^3}{3\sqrt{3}\beta} \cong 0.384\frac{h^3}{\beta}\,.$$

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The implicit scheme [1], Eq.1.96

$$
\begin{aligned}
\frac{u_j^{n+1}-u_j^n}{\tau} &+ \frac{\alpha}{4h}\left[u_j^n\left(u_{j+1}^{n+1}-u_{j-1}^{n+1}\right)+u_j^{n+1}\left(u_{j+1}^n-u_{j-1}^n\right)\right]+\\
&+\frac{\beta}{4h^3}\left(\left(u_{j+2}^{n+1}-2u_{j+1}^{n+1}+2u_{j-1}^{n+1}-u_{j-2}^{n+1}\right)+\right.\\
&\qquad\left.+\left(u_{j+2}^n-2u_j^{n+1}+2u_{j-1}^n-u_{j-2}^n\right)\right)=0\,.
\end{aligned}
\tag{3}
$$

• **Classical schemes with $\mathcal{O}(\tau^2,h^4)$ approximation** • The explicit scheme ([1], Eq.1.82)

$$
\begin{aligned}
u_i^{n+1} =& u_i^{n-1}-\frac{\alpha\tau}{6h}u_i^n\left(u_{i+2}^n-8u_{i+1}^n+8u_{i-1}^n-u_{i-2}^n\right)\\
&-\frac{\beta\tau}{4h^3}\left(u_{i+3}^n-8u_{i+2}^n+13u_{i+1}^n-13u_{i-1}^n+8u_{i-2}^n-u_{i-3}^n\right)
\end{aligned}
\tag{4}
$$

whose stability condition is given by

$$
\tau \le \frac{108h^3}{(43+7\sqrt{73})\sqrt{10\sqrt{73}-62}\beta} \cong 0.216\frac{h^3}{\beta}\,.
$$

The implicit scheme [1], Eq.1.84

$$
\begin{aligned}
\frac{u_j^{n+1}-u_j^n}{\tau} =& \frac{\alpha}{4h}\left[u_j^n\left(u_{j+2}^{n+1}-8u_{j+1}^{n+1}+8u_{j-1}^{n+1}-u_{j-2}^{n+1}\right)+\right.\\
&\left.+u_j^{n+1}\left(u_{j+2}^n-8u_{j+1}^n+8u_{j-1}^n-u_{j-2}^n\right)\right]+\\
&+\frac{\beta}{4h^3}\left(\left(u_{j+3}^{n+1}-8u_{j+2}^{n+1}+13u_{j+1}^{n+1}-13u_{j-1}^{n+1}+8u_{j-2}^{n+1}-u_{j-3}^{n+1}\right)+\right.\\
&\left.\left(u_{j+3}^n-8u_{j+2}^n+13u_{j+1}^n-13u_{j-1}^n+8u_{j-2}^n-u_{j-3}^n\right)\right)\,.
\end{aligned}
\tag{5}
$$

• **A new scheme with $\mathcal{O}(\tau^2,h^2)$ approximation** • It is obtained by the straightforward extension of the approach of paper [3] to equation (1). First, we convert (1) into the integral form

$$
\oint_\Gamma\left(-\frac{\alpha}{2}u^2-\beta u_{xx}\right)dt+u\,dx=0 \tag{6}
$$

valid for any simply connected integration contour Γ. Second, we chose the rectangular integration contour shown in Fig. stencil as a "control volume". Then we

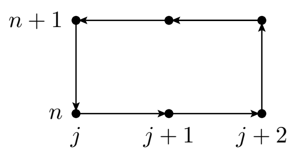

FIGURE 1. First integration contour for Eq. (6).

add to (6) the integral relations

$$\int_{x_j}^{x_{j+1}} u_x\,dx = u(t,x_{j+1}) - u(t,x_j),\ \int_{x_j}^{x_{j+2}} u_{xx}\,dx = u_x(t,x_{j+2}) - u_x(t,x_j). \quad (7)$$

To approximate the contour integral, we apply the trapezoidal rule. For numerical approximations of the integral relations we apply the trapezoidal rule for the integration of u_x and the midpoint rule for the integration of u_{xx}. Thereby, Eqs. (6) and (7) take the form

$$\begin{gathered}
\Big[-\frac{\alpha}{2}\left({u^2}^n_j + {u^2}^{n+1}_j - {u^2}^n_{j+2} - {u^2}^{n+1}_{j+2}\right) - \\
-\beta\left({u_{xx}}^n_j + {u_{xx}}^{n+1}_j - {u_{xx}}^n_{j+2} - {u_{xx}}^{n+1}_{j+2}\right)\Big]\cdot\frac{\tau}{2} + \left[u^{n+1}_{j+1} - u^n_{j+1}\right]\cdot 2h = 0, \\
\left[{u_x}^n_{j+1} + {u_x}^n_j\right]\cdot\frac{h}{2} = u^n_{j+1} - u^n_j,\, {u_{xx}}^n_{j+1}\cdot 2h = {u_x}^n_{j+2} - {u_x}^n_j.
\end{gathered} \quad (8)$$

To use linear difference elimination of u_x and u_{xx} from system (8), and hence the Maple package LDA (Linear Difference Algebra) [4], we introduce the new function $F := \dfrac{\alpha}{2}u^2$ and chose an elimination ranking $\succ$ such that $u \succ F \succ u_x \succ u_{xx}$. Then computation of a differenceGröbner basis of the ideal generated by the left-hand sides of difference polynomials in (8) and extraction from the basis an equation that does not contains u_x and u_{xx} yields the following difference scheme

$$\begin{gathered}
\frac{u^{n+1}_j - u^n_j}{\tau} + \frac{\alpha}{8h}\left[\left({u^2}^{n+1}_{j+1} - {u^2}^{n+1}_{j-1}\right) + \left({u^2}^n_{j+1} - {u^2}^n_{j-1}\right)\right] + \\
+\frac{\beta}{4h^3}\left[\left(u^{n+1}_{j+2} - 2u^{n+1}_{j+1} + 2u^{n+1}_{j-1} - u^{n+1}_{j-2}\right) + \right. \\
\left. + \left(u^n_{j+2} - 2u^{n+1}_j + 2u^n_{j-1} - u^n_{j-2}\right)\right] = 0.
\end{gathered} \quad (9)$$

This scheme is an analog of the famous Crank-Nicolson scheme for the heat equation.

• **A new scheme with $\mathcal{O}(\tau^2, h^2)$ approximation** • Choose now the integration contour Γ in (6) shown in Fig. 2 with the indentation $h/4$ along to the $x-$direction.

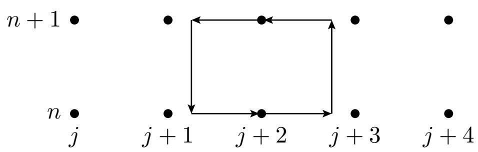

FIGURE 2. Second integration contour for Eq. (6).

Using at a fractional point relations

$$\begin{aligned}
f_{j+5/4} &\approx f_{j+1} + \frac{f_{j+2} - f_j}{2h}\cdot\frac{1}{4h} = (f_{j+2} + 8f_{j+1} - f_j)/8, \\
f_{j+11/4} &\approx f_{j+3} - \frac{f_{j+4} - f_{j+2}}{2h}\cdot\frac{1}{4h} = (-f_{j+4} + 8f_{j+3} + f_{j+2})/8.
\end{aligned} \quad (10)$$

we rewrite (6), (7) in the following form

$$\Big[-\frac{\alpha}{2}\Big(({-u^2}_j^n + 8{u^2}_{j+1}^n + {u^2}_{j+2}^n)/8 + ({-u^2}_j^{n+1} + 8{u^2}_{j+1}^{n+1} + {u^2}_{j+2}^{n+1})/8 -$$
$$- ({u^2}_{j+2}^n + 8{u^2}_{j+3}^n - {u^2}_{j+4}^n)/8 - ({u^2}_{j+2}^{n+1} + 8{u^2}_{j+3}^{n+1} - {u^2}_{j+4}^{n+1})/8\Big) -$$
$$- \beta\,\big((-{u_{xx}}_j^n + 8{u_{xx}}_{j+1}^n + {u_{xx}}_{j+2}^n)/8 + (-{u_{xx}}_j^{n+1} + 8{u_{xx}}_{j+1}^{n+1} + {u_{xx}}_{j+2}^{n+1})/8 -$$
$$- ({u_{xx}}_{j+2}^n + 8{u_{xx}}_{j+3}^n - {u_{xx}}_{j+4}^n)/8 - ({u_{xx}}_{j+2}^{n+1} + 8{u_{xx}}_{j+3}^{n+1} - {u_{xx}}_{j+4}^{n+1})/8)\big)\Big] \cdot \frac{\tau}{2} +$$
$$+ \left[u_{j+1}^{n+1} - u_{j+2}^n\right] \cdot \frac{3h}{2} = 0\,.$$
$$\left[{u_x}_{j+1}^n + {u_x}_j^n\right] \cdot \frac{h}{2} = u_{j+1}^n - u_j^n,\;\; {u_{xx}}_{j+1}^n \cdot 2h = {u_x}_{j+2}^n - {u_x}_j^n. \tag{11}$$

Elimination the grid functions u_x and u_{xx} from (11) gives the difference scheme

$$\begin{aligned} &\frac{u_j^{n+1} - u_j^n}{\tau} - \frac{\alpha}{48h}\Big[\Big({u^2}_{j+2}^{n+1} - 8{u^2}_{j+1}^{n+1} + 8{u^2}_{j-1}^{n+1} - {u^2}_{j-2}^{n+1}\Big) + \\ &\qquad + \Big({u^2}_{j+2}^{n} - 8{u^2}_{j+1}^{n} + 8{u^2}_{j-1}^{n} - {u^2}_{j-2}^{n}\Big)\Big] + \\ &-\frac{\beta}{16h^3}\left[\left(u_{j+3}^{n+1} - 8u_{j+2}^{n+1} + 13u_{j+1}^{n+1} - 13u_{j-1}^{n+1} + 8u_{j-2}^{n+1} - u_{j-3}^{n+1}\right) + \right. \\ &\qquad \left.\left(u_{j+3}^{n} - 8u_{j+2}^{n} + 13u_{j+1}^{n} - 13u_{j-1}^{n} + 8u_{j-2}^{n} - u_{j-3}^{n}\right)\right] = 0 \end{aligned} \tag{12}$$

Numerical results

Our numerical analysis of schemes (2)–(5), (9) and (12) was done with the Python package SciPy (`http:\\scipy.org`). as a benchmark, we used the exact one-soliton solution to (1)

$$u(x,t) = \frac{2k_1^2}{\cosh(k_1(x - 4k_1^2 t))^2}$$

with $\alpha = 6$, $\beta = 1$ and $k_1 = 0.4$. In so doing, we fixed $h = 0.25$ and considered the solution in interval $-50 \le x \le 50$ with periodic boundary conditions (cf. [1], p.49). The numerical inaccuracy was estimated by the Frobenius norm ($\|A\|_F$).

For the implicit schemes (9) and (12) we applied linearization

$$v_{k+1}^2 = v_{k+1}^2 - v_k^2 + v_k^2 = (v_{k+1} - v_k)(v_{k+1} + v_k) + v_k^2 \approx v_{k+1} \cdot 2v_k - v_k^2.$$

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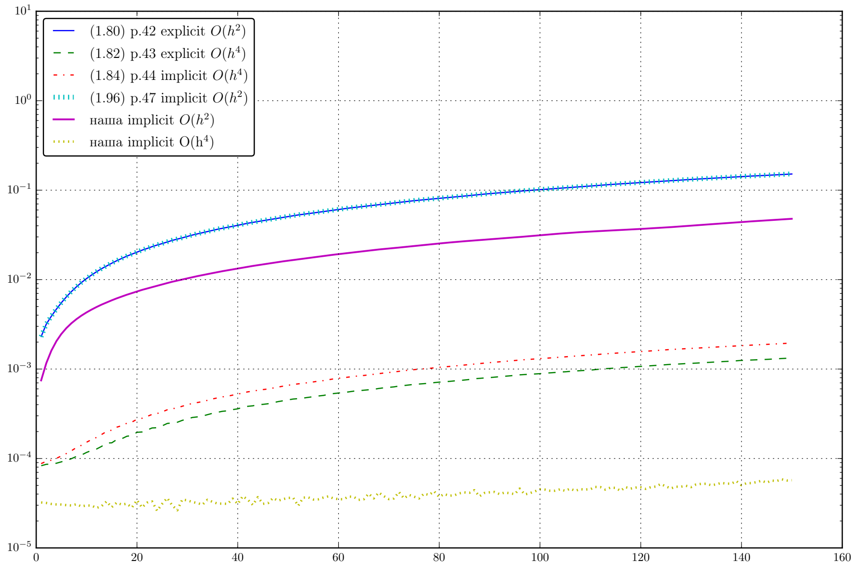

FIGURE 3. Dynamics of numerical error

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