On parametrization of nilpotent orbits of $Sp(N,\mathbb{C})$.

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Let $< \ldots, \ldots >$ be a non-degenerated scalar product, and $\mathfrak{G}$ be a matrix group preserving this product. Matrix A belongs to the corresponding Lie algebra $\mathfrak{g}$ iff $< A\xi, \eta > + < \xi, A\eta >= 0 \; \forall \xi, \eta$. Let ξ_1 and ξ_2 be eigenvectors corresponding to λ_1 and λ_2 correspondingly:

$$(\lambda_1 + \lambda_2) < \xi_1, \xi_2 >=< A\xi_1, \xi_2 > + < \xi_1, A\xi_2 >= 0.$$

Eigenvectors ξ_1 and ξ_2 are orthogonal if $\lambda_1 + \lambda_2 \neq 0$, particularly all eigenspaces corresponding non-zero eigenvalues are isotropic. Evidently $< (A - \lambda'\mathrm{I})\xi, \eta >= - < \xi, (A + \lambda'\mathrm{I})\eta >$ for any λ', consequently these two quadratic forms have the tame rank and if λ' is an eigenvalue , $-\lambda'$ is the eigenvalue too and $\ker^{\perp}(A - \lambda'\mathrm{I}) = \mathrm{im}(A + \lambda'\mathrm{I})$.

The method of the parametrization of the orbit $\mathcal{O}_J := \bigcup\limits_{g \in \mathfrak{G}} gJg^{-1}$ is based on the observation that (almost) any isotropic subspace can be transformed to the standard (coordinate) isotropic subspace of the proper dimension by an element of $\mathfrak{G}$

$$\Phi = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ \phi & \mathrm{I} & 0 \\ \phi_\Box & \tilde{\phi} & \mathrm{I} \end{pmatrix},$$

in the basis with the antidiagonal Gram-matrix of $< \ldots, \ldots >$. Such Φ transform A to the form

$$A = \Phi \begin{pmatrix} \lambda'\mathrm{I} & \rho & \rho_\Box \\ 0 & \widetilde{A} & \widetilde{\rho} \\ 0 & 0 & -\lambda'\mathrm{I} \end{pmatrix} \Phi^{-1},$$

and Lie-Poisson-Kirillov-Kostant forms calculated for A and for $\widetilde{A}$ differ on the value

$$\Delta\omega = \mathrm{tr}\, d(2\rho + \rho_\Box \phi^{\vdash} J) \wedge d\phi + \mathrm{tr}\, d\rho_\Box \wedge d\phi_\Box,$$

where $\phi^{\vdash}$ is the conjugation with respect to the antidiagonal[1] and $J = \mathrm{I}$ for the orthogonal groups and $J = diag(-\mathrm{I}, \mathrm{I})$ for the symplectic groups.

[1] $\phi^{\vdash} := \tau_1 \phi^T \tau_2$ where $\tau_{1,2}$ are the matrices of the inversion of the proper sizes

It is not difficult to see that if $\lambda' \neq 0$, the values ρ, ϕ and (anti)symmetrical parts of $\phi_\Box$ and $\rho_\Box$ are independent, consequently $P := 2\rho + \rho_\Box \phi^\vdash J, Q := \phi$ and over-antidiagonal parts of $\phi_\Box \pm \phi_\Box^\perp$ and $\rho_\Box$ form the corresponding part of the Darboux coordinates on the orbit. The zero eigenvalue is the complicated case. The eigenspace is not isotropic and this method does not work.

The theory can be adopted not for the eigenspace itself but for its isotropic subspaces. Such subspaces always exist, because $\ker A \cap \operatorname{im} A \neq 0$ and any subspace of this space is isotropic. All the speculations and the finale formula are the same. Orthogonal (or symplectic) transformation Φ moves the coordinate subspaces to the isotropic subspace of the kernel of A. The operation can be iterated, but the serious problem arise – the matrices ρ and $\rho_\Box$ *are not independent* now, $\rho = \rho(\widetilde{A})$ and $\rho_\Box = \rho_\Box(\widetilde{A})$.

The corresponding equations can be solved in one special case, when the isotropic space that we use for the construction of Φ is the smallest one, i.e. if we use $\ker A \cap \operatorname{im} A^{M-1} \neq 0$ such that $A^M = 0$. In this case the careful consideration of the Jordan forms of $A|_{\operatorname{im} A^k} / \ker A^m$ gives the following theorem.

Let the next flight of the iteration that transform the coordinate subspace to the smallest isotropic subspace of $\widetilde{A}$ gives the following matrices $\psi, \psi_\Box, \mu, \mu_\Box$:

$$\widetilde{A} = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ \psi & \mathrm{I} & 0 \\ \psi_\Box & \tilde{\psi} & \mathrm{I} \end{pmatrix} \begin{pmatrix} 0 & \mu & \mu_\Box \\ 0 & \tilde{\tilde{A}} & \widetilde{\mu} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 & 0 \\ \psi & \mathrm{I} & 0 \\ \psi_\Box & \tilde{\psi} & \mathrm{I} \end{pmatrix}^{-1},$$

their matrix elements are some rational functions of the matrix elements of $\tilde{A}$ (and A of cause).

Theorem. *On the algebraically open set of the matrix elements of A, the equations connecting $\phi, \phi_\Box, \rho, \rho_\Box$ and $\widetilde{A}$ are*

$$\rho \begin{pmatrix} \mathrm{I} \\ \psi \\ \psi_\Box \end{pmatrix} = 0,$$

here $\psi = \psi(\tilde{A})$, $\psi_\Box = \psi_\Box(\tilde{A})$.

It is $\zeta_M \times \zeta_{M-1}$ equations, where $\zeta_M = \dim \ker A \cap \operatorname{im} A^{M-1}$ is the number of the largest Jordan blocks $(M \times M)$ and $\zeta_{M-1} = \dim \ker A \cap \operatorname{im} A^{M-2} - \dim \ker A \cap \operatorname{im} A^{M-1}$ is the number of the Jordan blocks one unit smaller $((M-1)\times(M-1))$.

It is evident that the coordinate set is formed by the independent elements of all introduced matrix-blocks, i.e. all constructed matrices except the left $\zeta_k \times \zeta_{k-1}$ blocks of each constructed block-row (ρ, μ etc.). The *problem is to construct Darboux coordinates*, that is to combine all the independent variables in such a way that $\omega = \sum_i dp_i \wedge dq_i$. The problem is the cubic term $\operatorname{tr} d(\rho_\Box \phi^\vdash J) \wedge d\phi$. If both terms $d\rho \wedge d\phi$ and $d(\rho_\Box \phi^\vdash J) \wedge d\phi$ present, the sum has no necessary form. Let us consider the case when $\rho = 0$, it is the case $A^2 = 0$, there are the Jordan blocks 2×2 and eigenvectors without generalized eigenvectors. We consider the symplectic case $J = diag(-\mathrm{I}, \mathrm{I})$, the orthogonal case needs some technical work.

If $A^2 = 0$ we have only one step of the iteration:

$$\omega = \operatorname{tr} d(\rho_\Box \phi^\vdash J) \wedge d\phi + \operatorname{tr} d\rho_\Box \wedge d\phi_\Box.$$

It is known that a matrix from the Lie algebra $sp(N)$ has an even number of the eigenvectors without the generalized eigenvectors (the Jordan blocks of the odd size 1×1). It means that matrix $\phi^\vdash$ has even number of columns, and ϕ has even number of rows. Let us denote

$$\phi = \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix}, \quad \phi^\vdash = (\phi_+^\vdash, \phi_-^\vdash).$$

$$\omega = -\operatorname{tr} d(\rho_\Box \phi_+^\vdash) \wedge d\phi_- + \operatorname{tr} d(\rho_\Box \phi_-^\vdash) \wedge d\phi_+ + \operatorname{tr} d\rho_\Box \wedge d\phi_\Box =$$
$$= \operatorname{tr} d(2\rho_\Box \phi_-^\vdash) \wedge d\phi_+ + \operatorname{tr} d\rho_\Box \wedge d(\phi_\Box - \phi_+^\vdash \phi_-).$$

We use nice identity

$$\operatorname{tr} d(AB) \wedge dC + \operatorname{tr} d(BC) \wedge dA + \operatorname{tr} d(CA) \wedge dB = 0,$$

and the symmetry $\rho_\Box^\vdash = \rho_\Box$. The conjugated coordinates are symmetric matrix elements of $2\rho_\Box \phi_-^\vdash$ and ϕ_+ and the symmetric matrix elements of the over-antidiagonal elements of $\rho_\Box$ and $\phi_\Box - \phi_+^\vdash \phi_- + (\phi_\Box - \phi_+^\vdash \phi_-)^\vdash$, and the corresponding antidiagonal elements of square matrices $\rho_\Box$ and $\phi_\Box - \phi_+^\vdash \phi_-$.

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