Global Parametrizations and Local Expansions of One Real Variety with Boundary

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Abstract. An algebraic variety Ω in $\mathbb{R}^3$ is studied that plays an important role in the investigation of the normalized Ricci flow on generalized Wallach spaces related to invariant Einstein metrics. A procedure for obtaining a set of global parametric representations of the variety Ω is described, which is based on the use of the intersection of this variety with the discriminant set of an auxiliary cubic polynomial as the axis of parameterization. For this purpose, elimination theory and computer algebra are used. Obtained parametrizations are not valid at the finite set of a parameter and in this case we provide local expansions of the variety near its singular points.

Introduction

A three-parameter family of special homogeneous spaces was studied from the viewpoint of the normalized Ricci flow. In this case, the Ricci flow determines the evolution of invariant (homogeneous) Riemann metrics on the homogeneous Wallach spaces. The equation of the normalized Ricci flow is reduced to a system of two ordinary differential equations with three parameters (see [1] and the references in it for details), and the singular points of this system are associated with invariant Einstein metrics.

We investigate a set Ω, which is defined by zeroes of the polynomial $Q^*(a_1, a_2, a_3) \equiv Q(s_1, s_2, s_3) = 0$, where

$$\begin{aligned} Q \stackrel{\text{def}}{=} &(2s_1 + 4s_3 - 1)(64s_1^5 - 64s_1^4 + 8s_1^3 + 240s_1^2 s_3 - 1536 s_1 s_3^2 - \\ &- 4096 s_3^3 + 12 s_1^2 - 240 s_1 s_3 + 768 s_3^2 - 6 s_1 + 60 s_3 + 1) - \\ &- 8 s_1 s_2 (2s_1 + 4s_3 - 1)(2s_1 - 32 s_3 - 1)(10 s_1 + 32 s_3 - 5) - \\ &- 16 s_1^2 s_2^2 (52 s_1^2 + 640 s_1 s_3 + 1024 s_3^2 - 52 s_1 - 320 s_3 + 13) + \\ &+ 64(2s_1 - 1) s_2^3 (2s_1 - 32 s_3 - 1) + 2048 s_1 (2s_1 - 1) s_2^4. \end{aligned} \tag{1}$$

Here s_1, s_2, s_3 are the symmetric polynomials defined, respectively, as

$$s_1 = a_1 + a_2 + a_3, \quad s_2 = a_1a_2 + a_1a_3 + a_2a_3, \quad s_3 = a_1a_2a_3.$$

In [2] all singular points of Ω^* were found, and a qualitative description of all the components of it and their mutual positions was given.

A global parametrization of the variety can be useful in many cases:

- for the analysis of the intersection of a pair of varieties;
- for obtaining a local expansion of the variety at a point;
- for the visualization of varieties or their projections and so on.

In [3] a set of global rational parametrizations of the variety Ω was constructed.

Definition 1. An analytical description of the variety Ω in terms of the variables s_i is called an $\mathbf{s}$-representation, and in terms of the variables a_i an $\mathbf{a}$-representation.

1. Obtaining global parametrization of Ω

The procedure of obtaining global parametrization is described in details in [3]. Here we give the key points of it.

The zeroes of polynomial (1) provide the $\mathbf{s}$-representation of Ω only when the auxiliary cubic polynomial $\chi(y)$ with the coefficients $\mathbf{s}$

$$\chi(y) \stackrel{\text{def}}{=} y^3 - s_1y^2 + s_2y - s_3 \tag{2}$$

has three real zeroes.

Proposition 1. The polynomial $\chi(y)$ has only real searoes iff its coefficients $\mathbf{s}$ satisfy the inequality $D(\chi) \geq 0$, where

$$D(\chi) = -4s_1^3s_3 + s_1^2s_2^2 + 18s_1s_2s_3 - 4s_2^3 - 27s_3^2 \tag{3}$$

is the discriminant of (2).

The *discriminant surface* $\mathcal{D}(\chi) = \{\mathbf{s}\colon D(\chi) = 0\}$ divides the coefficient space Π of $\chi(y)$ into two domains and, as was shown in [4], it allows the polynomial parametrization

$$\mathcal{D}(\chi) : \{s_1 = 2t_1 + t_2,\ s_2 = t_1^2 + 2t_1t_2,\ s_3 = t_1^2t_2\}.$$

So, to get the parametrization of the variety Ω^* in $\mathbf{a}$ variables one has to compute the parametrization of those part of the variety Ω in $\mathbf{s}$, where polynomial $\chi(y)$ has three real zeroes, i.e. to compute parametrization for real variety with boundary.

The variety Ω intersects the discriminant surface $\mathcal{D}(\chi)$ along three one-dimensional varieties denoted by $\mathcal{Z}_i$, $i = 1, 2, 3$. These varieties are rational curves in the parameter space Π with parametrizations

$$\mathcal{Z}_1 : \left\{ s_1 = 2t_1 - 1/2, \qquad s_2 = t_1^2 - t_1, \qquad s_3 = -t_1^2/2 \right\},$$

$$\mathcal{Z}_2 : \left\{ s_1 = -\frac{u^2+u+1}{2u}, \quad s_2 = \frac{1-4u^2(u+1)^2}{4u}, \quad s_3 = \frac{(2u^2+2u+1)^2(u+1)}{32u^2} \right\},$$

$$\mathcal{Z}_3 : \left\{ s_1 = \frac{16u^3-1}{2\left(8u^2-1\right)}, \quad s_2 = \frac{u\left(8u^3-3u+1\right)}{8u^2-1}, \quad s_3 = -\frac{u\left(16u^3-4u+1\right)}{2\left(8u^2-1\right)} \right\}.$$

The varieties $\mathcal{Z}_i$, $i = 1, 2, 3$, intersect (or touch) at singular points $P_i^{(k)}$ of the set Ω (see [2, 3] for details) and they are shown in Fig. 1.

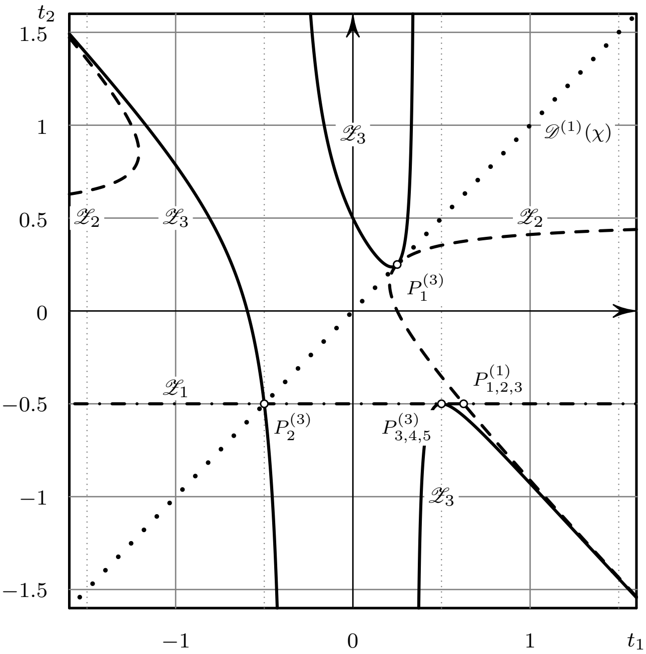

FIGURE 1. Varieties $\mathcal{Z}_i$, $i = 1, 2, 3$ and singular points $P_i^{(k)}$ on the plain (t_1, t_2).

The following approach for parametrization of the set Ω is proposed.

1. Choose one of the varieties $\mathcal{Z}_i$ and change the variables in such way that the new coordinates $\mathbf{S} = (S_1, S_2, S_3)$ give the deviation from this variety; let $\widetilde{Q}_i(\mathbf{S}) = 0$ be the equation of Ω in the new variables $\mathbf{S}$.
2. Determine the set $\mathbb{S}$ of the parameter's $S_1 = \text{const}$ values such that a curve $\mathcal{F} : \left\{ \widetilde{Q}_i(S_2, S_3) = 0 \right\}$ is reducible; such values are called *critical.*

3. Let for each fixed value of the parameter $S_1 \notin \mathbb{S}$ the irreducible algebraic curve $\mathcal{F}_i$ admits parametrization
$$\mathcal{F}_i : \{S_2 = \varphi(S_1, t), \quad S_3 = \psi(S_1, t)\}. \tag{4}$$
Parameter t is chosen such that for $t = 0$ vector $(S_1, \varphi(S_1, 0), \psi(S_1, 0))$ provides the parametrization of variety $\mathcal{Z}_i$.
4. Substitute parametrization (4) into (3) and find interval on which the inequality $D(S_1, t) \geq 0$ holds.

For each variety $\mathcal{Z}_i$, $i = 1, 2, 3$, the above procedure was applied and three rational parametrizations were obtained [3]. They are not valid for $s_1 \in \mathbb{S} \equiv \{-3/2, 0, 1/2, 3/4\}$ as at these values of parameter s_1 the polynomial (1) can be factorized into factors. Each of these factors gives the rational curve in the parameter space Π. Along each of these curves local expansions of the variety Ω were computed by algorithms of Power Geometry [5].

Finally, the parametrization of Ω^* in **a**-representation can be obtained with roots' formula of cubic equation for the *casus irreducibilis*.

References

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