Solving the Polynomial Equations by Algorithms of Power Geometry

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Abstract. New methods for computation of solutions of an algebraic equation of three variables near a critical point are proposed. These methods are: Newton polyhedron, power transformations, new versions of the implicit function theorem and uniformization of a planar algebraic curve. We begin from a survey of the new methods of computation of solutions of an algebraic equation of one and of two variables by means of the Hadamard polygon and Hadamard polyhedron. That approach works for differential equations (ordinary and partial) as well.

Introduction

We consider the polynomial equation $p(X) = 0$, where $X = (x_1, \dots, x_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$ and coefficients of p are from $\mathbb{R}$ or $\mathbb{C}$. We search its global solution in the whole space as well as its local solutions near its singular points X^0.

1. Global solutions

1a) If $n = 1$, solutions are several points. Using the Hadamard open polygon [1], [2, Part I, Ch. IV, Sect. 2.1], we obtain truncated equations $\check{p}_j(X) = 0$, $j = 1, \dots, m$, which are easily solved, and they give approximated roots of the initial polynomial $p(X)$. We can compute roots more precisely by the Newton method.

1b) If $n = 2$, solutions form an algebraic curve f. Sometimes solutions to the equation $p(X) = 0$ are known as functions $x_1 = \varphi(t)$, $x_2 = \psi(t)$. It is so, if the genus of the curve f is not big or if the polynomial $p(X)$ has rather big group of symmetries (i.e. birational automorphisms). Using the Hadamard polyhedron [3], we divide the space into m several pieces W_j, $j = 1, \dots, m$, and find the truncated simple polynomials $\check{p}_j(X)$, $j = 1, \dots, m$, which are the main parts of the initial polynomial $p(X)$ in the corresponding pieces W_j. Usually the truncated polynomial $\check{p}_j(X)$ has a lot of symmetries and it has an uniformization $x_1 = \varphi_j(t)$, $x_2 = \psi_j(t)$

of its roots. It gives the approximate parametrization of the curve f in the piece W_j. That parametrization can be made more accurate by the Newton method. So, we obtain m different approximate uniformizations in m pieces W_j.

1c) If $n > 2$ we can apply the similar method and can obtain several parametrizations in several pieces W_j of the whole space.

2. Local solutions

Near the critical point X^0 we can construct the Newton polyhedron [4] and can obtain several truncated polynomials $\hat{p}_j(X)$. Considering their solutions as the first approximations, we can continue them as the asymptotic expansions for each branch. Sometimes for that we need the global solutions of a polynomial equation with dimension $n' < n$ and transformations of the solutions into coordinate subspaces.

That approach works for *differential equations* (ordinary and partial) as well. In the survey [5] there are several nontrivial applications.

References

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