Riemann Surfaces and Branch Cuts

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Abstract. The problem of multiply valued functions in computer algebra systems is reviewed. This paper combines the two usual approaches to understanding such functions. The first approach describes the multiple values as *branches* in the complex plane. The second approach describes the multiple values as properties of the Riemann surface of the function. In this paper, a modified implementation in Maple allows us to work with these ideas more efficiently.

1. Introduction

The manner in which computer-algebra systems handle multivalued functions, specifically the elementary inverse functions, has been the subject of extensive discussions over many years. See, for example, [2, 4]. The discussion has centred on the best way to handle possible simplifications, such as

$$\sqrt{z^2} = z \ ? \quad \arcsin(\sin z) = z \ ? \quad \ln(e^z) = z \ ? \tag{1}$$

In the 1980s, errors resulting from the incorrect application of these transformations were common. Since then, systems have improved and now they usually avoid simplification errors, although the price paid is often that no simplification is made when it could be. For example, Maple 18 fails to simplify

$$\sqrt{1-z}\sqrt{1+z} - \sqrt{1-z^2} \ ,$$

even though it is zero for all $z \in \mathbb{C}$, see [2]. Here a new way of looking at such problems is presented.

The discussion of possible treatments has been made difficult by the many different interpretations placed on the same symbols by different groups of mathematicians. Sorting through these interpretations, and assessing which ones are practical for computer algebra systems, has been an extended process. In this paper, we shall not revisit in any detail the many past contributions to the discussion, but summarize them and jump to the point of view taken here.

1.1. A question of values

One question which has been discussed at length concerns the number of values represented by function names. One influential point of view was expressed by Carathéodory, in his highly regarded book [3]. Considering the logarithm function, he addressed the equation

$$\ln z_1 z_2 = \ln z_1 + \ln z_2 \ , \tag{2}$$

for complex z_1, z_2. He commented [3, pp. 259–260]:

> The equation merely states that the sum of one of the (infinitely many) logarithms of z_1 and one of the (infinitely many) logarithms of z_2 can be found among the (infinitely many) logarithms of $z_1 z_2$, and conversely every logarithm of $z_1 z_2$ can be represented as a sum of this kind (with a suitable choice of $\ln z_1$ and $\ln z_2$).

In this statement, Carathéodory first sounds as though he thinks of $\ln z_1$ as a symbol standing for a set of values, but then for the purposes of forming an equation he prefers to select one value from the set. Whatever the exact mental image he had, the one point that is clear is that $\ln z_1$ does not have a unique value, which is in strong contrast to every computer system. Every computer system will accept a specific value for z_1 and return a unique $\ln z_1$.

The reference book edited by Abramowitz & Stegun [1, Chap 4] is another authoritative source, as is its successor [9]. They both define, to take one example, the solution of $\tan t = z$ to be $t = \mathrm{Arctan}\, z = \arctan z + k\pi$. When listing properties, they both give the equation

$$\mathrm{Arctan}(z_1) + \mathrm{Arctan}(z_2) = \mathrm{Arctan}\,\frac{z_1 + z_2}{1 - z_1 z_2} \ . \tag{3}$$

For $z_1 = z_2 = \sqrt{3}$, Maple and Mathematica simplify this identity to $\pi/3 = -\pi/3$.

Riemann surfaces give a very pictorial way of seeing multi-valuedness [10, 5], but a question remains whether they can be used computationally [7]. Here we shall continue an approach to Riemann surfaces described by [5] and in references given therein.

2. A new treatment of inverse functions

The basis of the new implementation is notation first introduced in [6]. To the standard function $\ln z$, a subscript is added:

$$\ln_k z = \ln z + 2\pi i k \ .$$

Here the function $\ln z$ denotes the principal value of logarithm, which is the single-valued function with imaginary part $-\pi < \Im \ln z \le \pi$. This is the function currently implemented in Maple, Mathematica, Matlab and other systems. In contrast, $\ln_k z$ denotes the kth branch of logarithm. With this notation, the statement above of Carathéodory can be restated unambiguously as

$$\exists k, m, n \in \mathbb{Z}, \text{ such that } \ln_k z_1 z_2 = \ln_m z_1 + \ln_n z_2 \ .$$

His "and conversely" statement is actually a stronger statement. He states

$$\forall k \in \mathbb{Z}, \exists m, n \in \mathbb{Z}, \text{ such that } \ln_k z_1 z_2 = \ln_m z_1 + \ln_n z_2 \ .$$

In the light of his converse statement, Carathéodory's first statement could be interpreted as meaning

$$\forall m, n \in \mathbb{Z}, \exists k \in \mathbb{Z}, \text{ such that } \ln_m z_1 + \ln_n z_2 = \ln_k z_1 z_2 \ .$$

This shows the greater conciseness of branch notation.

3. An example function

To save space, we consider only the inverse sine function. The principal branch of the inverse sine function is denoted in Maple by `arcsin`. Using this, we define the branched inverse sine by

$$\operatorname{invsin}_0 z = \arcsin z \ , \tag{4}$$

$$\operatorname{invsin}_k z = (-1)^k \operatorname{invsin}_0 z + k\pi \ . \tag{5}$$

The principal branch now has the equivalent representation $\operatorname{invsin}_0 z = \operatorname{invsin} z = \arcsin z$. It has real part between $-\pi/2$ and $\pi/2$. Notice that the branches are spaced a distance π apart in accordance with the antiperiod[1] of sine, but the repeating unit is of length 2π in accord with the period of sine.

The Maple code for the function is

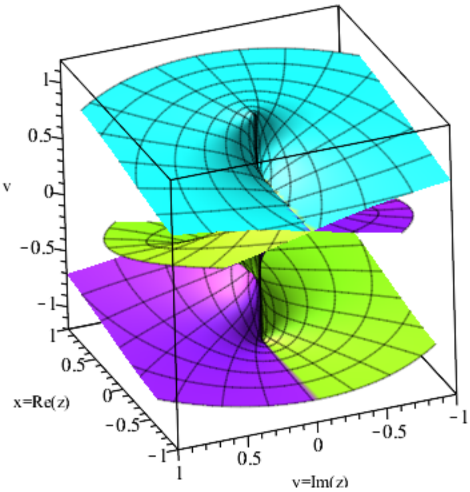
```
invsin := proc (z::algebraic) local branch;
             if nargs <> 1 then
                 error "Expecting 1 argument, got", nargs ;
             elif type(procname, 'indexed') then
                 branch := op(procname);
                 branch*Pi+(-1)^branch*arcsin(z);
             else arcsin(z);
             end if;
          end proc;
```

4. Riemann surfaces

Many multivalued functions are the inverses of single-valued functions. This allows the Riemann surface to be plotted easily. To plot the cube-root surface in Maple, we issue the plot command, which uses the branch notation above for colouring.

```
plot3d([Re((u+I*v)^3),Im((u+I*v)^3), u], u = -1 .. 1, v = -1 .. 1,
colour = (2+k3(u+I*v))*(1/4), view = [-1 .. 1, -1 .. 1, -1 .. 1])
```

[1] An antiperiodic function is one for which $\exists \alpha$ such that $f(z+\alpha) = -f(z)$, and α is then the antiperiod. This is a special case of a quasi-periodic function [8], for which $\exists \alpha, \beta$ such that $f(z+\alpha) = \beta f(z)$.



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