Quantum Behavior in the Framework of Permutation Groups

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Abstract. The trajectory of a quantum system is a sequence of unitary evolutions of vectors in a Hilbert space, interspersed with observations — projections of the vectors in some subspaces, that are specified by measuring devices. Quantum-mechanical description can be made constructive, if we replace the general group of unitary transformations of the Hilbert space by unitary representations of finite groups. It is known that any linear representation of a finite group can be realized as a subrepresentation of some permutation representation. Thus, quantum mechanical problems can be formulated in terms of groups of permutations. Such a constructive approach allows us to clarify the meaning of a number of physical concepts.

1. Quantum Mechanics. Briefly, the formalism of quantum mechanics is reduced to the following. A **pure quantum state** is a ray in a Hilbert space $\mathcal{H}$ over $\mathbb{C}$, i.e. an equivalence class of non-zero vectors in $\mathcal{H}$: $|\psi\rangle \sim a\,|\psi\rangle$, $a \in \mathbb{C}$. The normalization $\langle\psi\,|\,\psi\rangle = 1$ reduces the equivalence: $|\psi\rangle \sim \mathrm{e}^{\mathrm{i}\alpha}\,|\psi\rangle$, $\alpha \in \mathbb{R}$. The phase factor $\mathrm{e}^{\mathrm{i}\alpha}$ can be eliminated by replacing the normalized vector $|\psi\rangle$ with the rank one **projector** $\Pi_\psi = |\psi\rangle\langle\psi|$. A weighted mixture of pure quantum states is called a **mixed quantum state**. Any quantum state can be represented by a **density matrix** ρ with characteristic properties: $\rho = \rho^\dagger$, $\rho \geq 0$, $\operatorname{tr}\rho = 1$. In particular, for a pure state $|\psi\rangle$ the density matrix is the projector $|\psi\rangle\langle\psi|$. The eigenvalues $p_1, p_2, \ldots$ of a density matrix form a probability distribution in an ensemble of pure states. The Hilbert space of a **composite system** is the *tensor product* of Hilbert spaces for the subsystems: $\mathcal{H}_{XY} = \mathcal{H}_X \otimes \mathcal{H}_Y$. Any mixed state in a Hilbert space $\mathcal{H}$ can be obtained by taking partial trace of a pure state in a "larger" Hilbert space that contains $\mathcal{H}$ as a tensor factor (this is called *"purification"*). **Observation** is detection of the state of a quantum system in one of the mutually orthogonal subspaces that form a partition (defined by an observational setup) of the Hilbert space. The result of quantum observation is random and its statistics is described by a probability measure defined on the subspaces. **Gleason's theorem** states that (for $\dim\mathcal{H} > 2$) any suitable measure has the form $\mu_\rho(S) = \operatorname{tr}(\rho\Pi_S)$, where ρ is an arbitrary density matrix, Π_S is the orthogonal projection in the subspace $S \leq \mathcal{H}$. In the case of pure state, $\rho = |\varphi\rangle\langle\varphi|$, and one-dimensional subspace $S = \operatorname{span}\{|\psi\rangle\}$, we have the usual **Born rule**: $\operatorname{tr}(\rho\Pi_S) \equiv |\langle\varphi\,|\,\psi\rangle|^2$. **Measurement** is a particular case of observation when the orthogonal partition of a Hilbert space is formed by the eigenspaces of an *observable* (an arbitrary Hermitian operator) A. The eigenvalues of A are considered as measured values. The **expectation value** of A in

the state ρ is defined as $\langle A\rangle_\rho = \mathrm{tr}\,(\rho A)$. The **time evolution** of a quantum system between observations is described by a *unitary transformation*: $|\psi_{t'}\rangle = U_{t't}\,|\psi_t\rangle$, or $\rho_{t'} = U_{t't}\rho_t U_{t't}^\dagger$.

2. Constructive Modification of Quantum Formalism. To build constructive models, we need to remove infinities from the formalism. Formally, one can prove that the unitary group $\mathsf{U}(n) \cong \mathrm{Aut}\,(\mathcal{H}_n)$, which has the cardinality of continuum, is empirically equivalent to a **finite group** G. This means that for any particular problem it is always possible to pick a finite group G such that its unitary representation can replace $\mathsf{U}(n)$ without losing the accuracy of describing the empirical data. In essence, it seems more natural to assume that at the fundamental level, it is the finite groups that act, and the continuous unitary groups are only continuum approximations of their unitary representations.

The advantages of finite groups are:

- any finite group is a subgroup of a symmetric group,
- any linear representation of a finite group is unitary and is a subrepresentation of some permutation representation,
- from an empirical point of view, any continuous group can be approximated by a finite one, but not vice versa.

Natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ and **roots of unity** $\mathsf{r} \mid \mathsf{r}^k = 1$ are sufficient to represent all numbers that are significant in the quantum formalism. Let $\mathbb{N}\,[\mathsf{r}_k]$ denote the extension of the semiring $\mathbb{N}$ by a kth primitive root of unity. $\mathbb{N}\,[\mathsf{r}_k]$ is a ring for $k \geq 2$. In particular, this construction gives a way to introduce negative numbers: $\mathbb{Z} = \mathbb{N}\,[\mathsf{r}_2]$. The fraction field of the ring $\mathbb{N}\,[\mathsf{r}_k]$ is the kth **cyclotomic field** $\mathbb{Q}(\mathsf{r}_k)$, which is a dense subfield of $\mathbb{C}$ for $k \geq 3$.

Let G act by permutations on a finite domain Ω, $|\Omega| = \mathsf{N}$. This action induces a **permutation representation** of G on the module $\mathbb{N}^{\mathsf{N}}$ over the semiring $\mathbb{N}$. Any linear representation of G can be realized over a cyclotomic field $\mathbb{Q}\,(\mathsf{r}_k)$, where k is some divisor of the *exponent* of G. Let us extend the module $\mathbb{N}^{\mathsf{N}}$ to the Hilbert space $\mathcal{H}_{\mathsf{N}}$ by extending $\mathbb{N}$ to $\mathbb{Q}\,(\mathsf{r}_k)$. Any **constructive representation** of G can be obtained by projecting the permutation representation of G on the module $\mathbb{N}^{\mathsf{N}}$ (for some sufficiently large N) in an *invariant subspace* of the Hilbert space $\mathcal{H}_{\mathsf{N}}$.

3. Modeling of Quantum Evolution. The time evolution of a quantum system is a sequence of observations with unitary transitions between them. The role of observations is most impressively manifested in the *quantum Zeno effect*. Since the observation of even a pure state leads to a mixed state, it is natural to describe evolution in terms of the density matrix. Suppose that the fundamental ("Planck") time is a sequence of integers: $\mathcal{T} = \{\ldots, 0, 1, 2, \ldots\}$. We define "empirical time" as a sequence of "instants of observations": $t_0, t_1, \ldots, t_{i-1}, t_i, \ldots$. The simplest assumption is that the instants of observations are elements of the fundamental time: $t_i \in \mathcal{T}$. More realistic model of the empirical time would be a distribution around t_i, e.g. the binomial distribution

$$K_\sigma\,(\tau - t_i) = \frac{(2\sigma)!}{4^\sigma\,(\sigma - t_i + \tau)!\,(\sigma + t_i - \tau)!}, \qquad t_i - \sigma \leq \tau \leq t_i + \sigma. \tag{1}$$

At present, the smallest time uncertainty, i.e. the analog of σ, in physical experiments is about 10^{26} Planck time units. Let ρ_i denote the state of the system *after* the ith observation. According to the standard quantum mechanics, the state *before* the ith observation takes the form $\rho_i' = U\rho_{i-1}U^\dagger$, where $U = \mathrm{e}^{-\mathrm{i}H(t_i - t_{i-1})}$ is a single unitary evolution with a given Hamiltonian H. If ρ_{i-1} is pure, then ρ_i' is also pure. Hamiltonians are usually derived from the principle of least action. Motivated by the fact that any extremal principle involves a selection of dominant elements among many candidates, we propose the following modification: $\rho_i' = \sum_{k=1}^{\mathsf{M}} w_{ik} U_k \rho_{i-1} U_k^\dagger$, where $U_1, \ldots, U_{\mathsf{M}}$ are elements of a unitary representation of a finite group of size M, w_{ik} are (normalized: $\sum_{k=1}^{\mathsf{M}} w_{ik} = 1$) weights of the group elements at the transition $t_{i-1} \to t_i$. A trivial choice of weights reproduces the standard scheme, but in general ρ_i' is a mixed state. The single-step and N-step transition probabilities are $\mathbf{P}_i = \sum_{k=1}^{\mathsf{M}} w_{ik} \operatorname{tr}\left(U_k \rho_{i-1} U_k^\dagger \rho_i\right)$ and $\mathbf{P}_{0\to N} = \prod_{i=1}^{N} \mathbf{P}_i$, respectively. The single-step transition entropy $\Delta\mathbf{S}_i = -\log \mathbf{P}_i$ is a discrete analog of Lagrangian $\mathcal{L}$, and the corresponding entropy of trajectory $\mathbf{S}_{0\to N} = \sum_{i=1}^{N} \Delta\mathbf{S}_i$ is an analog of action $\mathcal{S} = \int \mathcal{L} dt$.

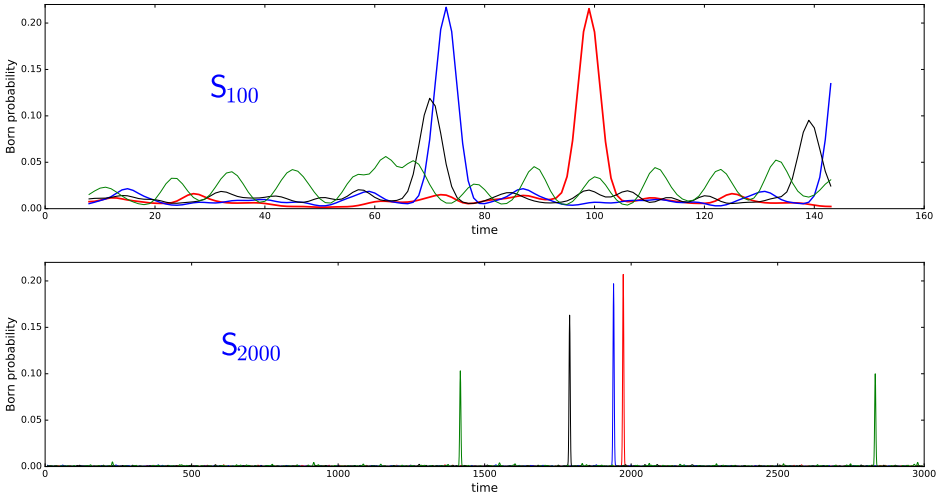
4. The Principle of Least Action as Continuum Approximation. To obtain the continuum limit of the above-described quantum evolution model, we proceed as follows. Replacing the finite group by a Lie group, we introduce the **Lie algebra approximation** in the vicinity of the identity element. For a matrix of unitary representation we have $U \approx \mathbb{1} + \mathrm{i}A$, where A is a Hermitian matrix. We use also the **linear approximation** for the differences of variables by introducing the time derivatives: $X_i - X_{i-1} \approx \dot{X}_i\,(t_i - t_{i-1})$. Discrete sequences are replaced by continuous functions. Note that the case of general mixed states does not allow reasonable continuous approximations: it is natural to assume that the probability of a transition between close density matrices should tend to unity, however $\operatorname{tr}\left(\rho^2\right) = 1$ implies that ρ is a projector. Thus, we shall assume here the case of pure states ψ. All this, together with some standard approximations, leads to the following expression for the Lagrangian $\mathcal{L} = \left\langle \psi \middle| \dot{A}^2 \middle| \psi \right\rangle - \left\langle \psi \middle| \dot{A} \middle| \psi \right\rangle^2 - \mathrm{i}\left(\left\langle \dot{\psi} \middle| \dot{A} \middle| \psi \right\rangle - \left\langle \psi \middle| \dot{A} \middle| \dot{\psi} \right\rangle + 2 \left\langle \psi \middle| \dot{A} \middle| \psi \right\rangle \left\langle \psi \middle| \dot{\psi} \right\rangle \right) - \left\langle \psi \middle| \dot{\psi} \right\rangle^2$.

5. Searching Dominant Unitary Evolutions in Natural and Standard Representations of Symmetric Group. The usual **natural representation** of S_N in the Hilbert space $\mathcal{H}_\mathsf{N}$ can be obtained from the permutations of coordinates in the module $H = \mathbb{N}^\mathsf{N}$ by extending the semiring $\mathbb{N}$ to the field $\mathbb{C}$. The natural representation decomposes into the one-dimensional **trivial** and $(\mathsf{N}-1)$-dimensional **standard** irreducible representations. In fact, any representation of S_N can be realized over $\mathbb{Q}$ (and over $\mathbb{Z}$). The inner product in the module H is defined as $\langle m \,|\, n \rangle = \sum_{k=1}^{\mathsf{N}} m_k n_k$ for $m, n \in H$. Let $|x\rangle$ denote the normalized natural vector: $|x\rangle = |n\rangle \,/ \sqrt{\langle n \,|\, n \rangle}$. For the natural pure states we have the usual Born rule $\mathbf{P}_{\text{nat}}\,(x, y) = \langle x \,|\, y \rangle^2$. The space of the standard representation is the $(\mathsf{N}-1)$-dimensional 'standard' subspace in $\mathcal{H}_\mathsf{N}$ defined by the condition $x_1 + \cdots + x_\mathsf{N} = 0$.

Any vector of the standard subspace can be expressed via the projection of some normalized natural vector. The Born probability in the standard subspace in terms of the normalized natural vectors takes the form

$$\mathbf{P}_{\mathrm{std}}\,(x,y) = \frac{(\mathsf{N}\,\langle x\,|\,y\rangle - ab)^2}{(\mathsf{N}-a^2)\,(\mathsf{N}-b^2)},\ \text{where } a = \sum_{k=1}^{\mathsf{N}} x_k \text{ and } b = \sum_{k=1}^{\mathsf{N}} y_k.$$

Consider the evolution of the initial vector $|x\rangle$, the result of which is measured by the vector $|y\rangle$. The dominant evolutions, i.e. those that provide maximum interaction with the observation device, are of the most interest, since they mainly determine the observed behavior of a quantum system. In the case of the natural and standard representations of the group S_N, all the dominant evolutions for any given pair of vectors $|x\rangle$ and $|y\rangle$ can be found by simple algorithms. The two figures below show the dominant evolutions for four randomly generated pairs of vectors $|x\rangle$ and $|y\rangle$ for the cases $\mathsf{N} = 100$ and $\mathsf{N} = 2000$. The graphs of time evolution of Born's probabilities are smoothed by the binomial kernel (1) with $\sigma = 7$. It is seen that with increasing N, the relative weight of dominant evolutions rapidly increases.



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