Two formulas of planetary motion

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Abstract. Together the first and the second Kepler's laws (of planetary motion) imply the third. Kepler's first law might be expressed via a polar equation of an ellipse representing the orbit, with the origin (of the coordinate system) representing the Sun. The second law might be combined with the third via an elementary formula which readily applies to calculating the length of the four seasons. Although such formula would not take into account the precession and nutation of Earth's axis but relies on a single parameter (that is, the eccentricity of the orbit), it coincides (up to a proper fraction of an hour) with the observable lengths of the seasons!

The first Kepler's law might be expressed via a formula, expressing the distance r between the Sun and its orbiting planet as a function of the true anomaly θ:

$$r = r(\theta) = \frac{p}{1 + e\cos\theta},\ p := \frac{b^2}{a} = a\left(1 - e^2\right),$$

where p is the (so-called) semilatus rectum, whereas a and b are the lengths of semi-major and semi-minor axes, respectively. The ellipticity of the orbit imposes upon its eccentricity e the condition: $-1 < e < 1$.

Kepler's equation calculates the (so-called) mean anomaly M via the eccentric anomaly E as

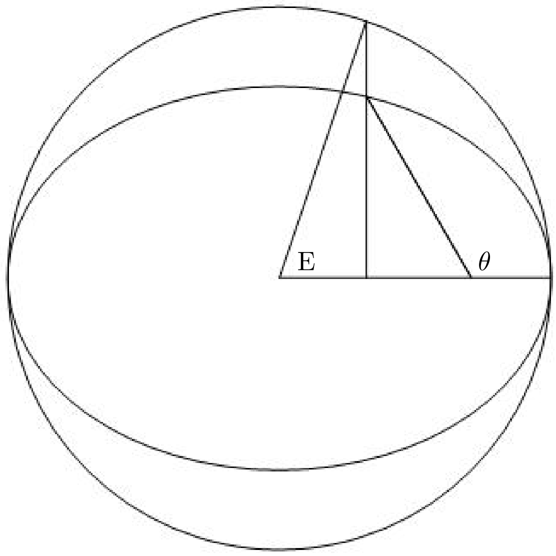
$$M = E - e\sin E.$$ [1]

We might, as well, calculate the mean anomaly M as a function of the true anomaly θ as

$$M = M(\theta) = 2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right) - \frac{e\sqrt{1-e^2}\,\sin\theta}{1 + e\cos\theta},$$

[1] At the end of the Fourth Part of his work "De Motibus Stellae Martis", Kepler states, according to a traslation from the Latin [1], concerning the solution of the problem so long known by his name (that is, concerning the determination of E for a given M):

I am sufficiently satisfied that it cannot be solved a priori, on account of the different nature of the arc and the sine. But if I am mistaken, and any one shall point out the way to me, he will be in my eyes the great Apollonius.



Eccentric and true anomalies

where the angle θ might be confined to lie in the closed interval $[-\pi, \pi]$ as the principal branch of the (multi-valued) function $\arctan(\cdot)$ is assumed. Thus, the two summands on the right hand side of the latter equation correspond to the two summands on the right hand side of the preceding Kepler's equation. One might verify the equivalence of the two formulas (for the mean anomaly M) via the accessory identities:

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E}, \ \sin\theta = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}, \ \tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right).$$

Now, the second and the third Kepler's laws might be unified (and strengthened) via a formula for the orbiting time $t = t(\theta)$ as

$$t = \sqrt{\frac{a^3}{\mu}}\, M, \tag{1}$$

where μ is the product of the mass of the Sun with the gravitational constant.

Differentiating the latter time function t with respect to the true anomaly θ and taking the reciprocal, we readily calculate $\dot{\theta}$, thereby deriving a strong version of Kepler's second law as

$$\dot{\theta}\, r^2 = \sqrt{\mu\, p},$$

where the dot above denotes differentiation with respect to time. Moreover, integrating the latter equality, over a full period T, readily yields Kepler's third law as

$$2\pi ab = \sqrt{\mu\, p}\, T \implies T = 2\pi\sqrt{\frac{a^3}{\mu}}.$$

Moreover, having explicitly expressed t as a function of θ, we might calculate the lengths of the four seasons on Earth. Assuming $e = 5/299$, $T = 1461/4$ and the value of the true anomaly θ at the vernal equinox is $3\pi/7$, we calculate the lengths of Winter and Spring as

$$t\left(\frac{3\pi}{7}\right) - t\left(-\frac{\pi}{14}\right) \approx 88.995,\ t\left(\frac{13\pi}{14}\right) - t\left(\frac{3\pi}{7}\right) \approx 92.765,$$

respectively. If we maintain the values of the argument θ but flip the sign of eccentricity ($e \mapsto -e$), then the two differences (upon evaluating t) would correspond to the lengths of Summer (93.651) and Autumn (89.839), respectively. Thus, the length of the "polar night" at the North (South) Pole is 178.83 (186.42).[2] These calculations closely agree with the actual lengths of seasons, which are subject to small alterations which order of magnitude matches the (20 minutes) discrepancy between the tropical and the sidereal year along with other lesser components of precession (and nutation) of the equinoxes, such as the Chandler wobble [2], which we shall further discuss at the upcoming talk.

Formula (1), where the mean anomaly M expressed via the true anomaly θ, is highly relevant for constructing "the rotating celestial sphere" [3], which pilot version was publicly first ever presented by Takayuki Ohira in Moscow, Russia, via a video communication from Yokohama, Japan, on April 2, 2017 [4].

References

[1] Hall A. Kepler's Problem // The Analyst **10** (3), May 1883: 65-66. Available at `http://www.jstor.org/stable/2635832`.

[2] Furuya M. & Chao B. F. *Estimation of period and Q of the Chandler wobble* // Geophysical Journal International **127**(3), 1996: 693-702.

[3] Adlaj S. *The rotating celestial sphere.* Available at `http://www.ccas.ru/depart/mechanics/TUMUS/Adlaj/CelSph.pdf`.

[4] Ohira T. A presentation of the rotating celestial sphere (on April 2, 2017) via a video communication at the First Open Junior Scientist Conference, Moscow, Russia. Available at `https://vk.com/public132056427`.

[2]The Winter at the North Pole does not get nearly as cold as the Summer gets at the South Pole, as consistently recorded and observed. The length of the "polar night" at the North Pole is over a week shorter than its length at the South Pole (although the small eccentricity of Earth's orbit makes it visibly indistinguishably from a circle). No temperature lower than that recorded at Mount McKinley, Alaska (−73.8 °C) was ever recorded at the Northern Hemisphere of Earth. Yet, the lowest temperature ever directly recorded at ground level on Earth is −89.2 °C (184.0 K), which occurred at the Soviet Vostok Station in Antarctica on July 21, 1983.

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