On the Restriction of Smooth Plane Sextics

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Abstract. A double cover of the projective plane branched along a non-singular sextic curve is a genus 2 $K3$ surface. In this regard, the restriction of parameters of defining equation of non-singular sextic curve is an interesting object. We consider about the restriction of the defining equation that the dimension of the parameter is equal to 19.

1. Introduction

1.1. Sextic curves

Let $\mathbb{P}^2$ be a 2-dimensional complex projective space with the coordinate $[x, y, z]$ and let $f_6(x, y, z)$ be a homogeneous polynomial of variables x, y, z with degree 6 in $\mathbb{P}^2$. We consider the set

$$V_6 := \{(x, y, z) | f_6(x, y, z) = 0\}.$$

We call V_6 complex projective plane degree 6 curves (later, we call them sextic curves). The dimension of parameters of defining equation of non-singular sextic curve is less than or equal to 19 ([4]).

1.2. Singular point

Let $f(x, y, z)$ be a homogeneous polynomial of variables x, y, z with degree 6 in $\mathbb{C}^3$. Then

$$f(0,0,0) = \frac{\partial f(0,0,0)}{\partial x} = \frac{\partial f(0,0,0)}{\partial y} = \frac{\partial f(0,0,0)}{\partial z} = 0.$$

Hence, the analytic set defined by $f(x, y, z) = 0$ has a singular point at the origin in $\mathbb{C}^3$. The analytic set is a non-singular sextic curve in $\mathbb{P}^2$ if it has only isolated singular point at the origin in $\mathbb{C}^3$ (later, we call them smooth plane sextics). For a defining equation of an analytic set which has an isolated singular point at the origin in $\mathbb{C}^3$, there exist the following theorem([1]).

Theorem 1.1 (Arnold). *Let $f(z_0, z_1, z_2)$ be a polynomial in $\mathbb{C}^3$ and let V be an anlytic set such that $V = \{(z_0, z_1, z_2) | f(z_0, z_1, z_2) = 0\}$ which has an isolated singular point at the origin in $\mathbb{C}^3$. Then, for any i ($i = 0, 1, 2$), there exists an integer $a_i \geq 1$ and f has a term $z_i^{a_i} z_j$.*

2. Elimination ideal and restriction

Theorem 2.1 (The Elimination Theorem [3]). *Let $I \subset k[x_1, \dots, x_n]$ be an ideal and let G be a Gröbner basis of I with respect to the lex order where $x_1 > x_2 > \dots > x_n$. Then, for every $0 \le l \le n$, the set*

$$G_l = G \cap k[x_{l+1}, \ \dots \ , x_n]$$

is a Gröbner basis of the l-th elimination ideal I_l.

Theorem 2.2 (The Extension Theorem [3]). *Let $I = \langle f_1, \dots, f_s \rangle \subset \mathbb{C}[x_1, \dots, x_n]$ and let I_l be the first elimination ideal of I. For each $1 \le i \le s$, write f_i in the form*

$$f_i = g_i(x_2, \dots, x_n)x_1^{N_i} + \textit{terms in which } x_i \textit{ has degree} < N_i \ ,$$

where $N_i \ge 0$ and $g_i \in \mathbb{C}[x_2, \dots, x_n]$ is nonzero. Suppose that we have a partial solution $(a_2, \dots, a_n) \in V(I_l)$. If $(a_2, \dots, a_n) \notin V(g_1, \dots, g_s)$, then there exists $a_1 \in \mathbb{C}$ such that $(a_1, a_2, \dots, a_n) \in V(I)$.

Let f be a defining equation, $I := \langle \ f, \dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \dots, \dfrac{\partial f}{\partial x_n} \ \rangle$. Then we can obtain the non-degeneracy condition for the singularity at the origin from the Gröbner basis. We call this non-degeneracy condition the restriction([1]). We need to classify the conditions for the parameters of the leading term. We consider the ideal $I = \{f_i(t_1, \dots, t_m, x_1, \dots, x_n) : 1 \le i \le s\}$ in $k(t_1, \dots, t_m)[x_1, \dots, x_n]$ and fix a monomial order. Here, $t_1, \dots, t_m$ are symbolic parameters appearing in the coefficients of $f_1, \dots, f_s$. We can divide each f_i by its leading coefficient in $k(t_1, \dots, t_m)$, under the assumption that these leading coefficients are equal to 1. Then let $g_1, \dots, g_s$ be a reduced Gröbner basis for I, and the leading coefficients of g_i are then also 1 ([3]).

We consider a plane cubic in $\mathbb{C}^3$ as an example. Let $f = x^2z + y^3 + pyz^2 + qz^3$, then the set defined by $f = 0$ is a cubic curve in $\mathbb{C}^3$. If the cubic curve has an isolated singularity at the origin, the only solution of $(f, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}) = (0,0,0,0)$ is $x = y = z = 0$. Let $f_1 = \dfrac{\partial f}{\partial x}$, $f_2 = \dfrac{\partial f}{\partial y}$, $f_3 = \dfrac{\partial f}{\partial z}$ and let G be the Gröbner basis of f, f_1, f_2, f_3. Then the Gröbner basis of the 3rd elimination ideal, I_3, is

$$G_3 = G \cap \mathbb{C}[p, \ q] = \phi.$$

And

$$G \cap \mathbb{C}[x,p,q] = x^3,$$
$$G \cap \mathbb{C}[y,p,q] = (4p^3 + 27q^2)y^4,$$
$$G \cap \mathbb{C}[z,p,q] = (4p^3 + 27q^2)z^4.$$

Hence, we obtain $4p^3 + 27q^2 = 0$ as the degeneracy condition of f. We consider the defining equation that has singularity and is constrained by the Milnor number of singularity. We call this condition the restriction of the definition equation's parameter, In this example, f is constrained by restriction $4p^3 + 27q^2 \neq 0$.

3. Restriction

We consider about only the defining equation that the dimension of the parameter is equal to 19. We got the restrictions on smooth plane quartics by using the Gr:obner basis. We use the method([5]). Then, for the smooth plane sextics, the following result holds.

Result 3.1. *In $\mathbb{P}^2$ with the coordinate $[x, y, z]$, let f be the defining equation that the dimension of the parameters of smooth plane sextics is equal to 19. Then f is as follows..*
$f = x^5z + (y^3 + a_1yz^2 + a_2z^3)x^3 + (a_3y^4 + a_4y^3z + a_5y^2z^2 + a_6yz^3 + a_7z^4)x^2$
$+ (a_8y^5 + a_9y^4z + a_{10}y^3z^2 + a_{11}y^2z^3 + a_{12}yz^4 + a_{13}z^5)x$
$+ a_{14}y^6 + a_{15}y^5z + a_{16}y^4z^2 + a_{17}y^3z^3 + a_{18}y^2z^4 + a_{19}yz^5 + a_{20}z^6$
where a_i is parametric coefficient ($1 \leq i \leq 20$).

The result of the calculation of the elimination ideal becomes the key to solve the problem of the restriction of the parameters.

Changing the coordinates so that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$.

Then the defining equation is as follows (We rewrite x', y', z' to x, y, z again after having transformed them) :
$f = x^5z + g_1x^4z^2 + (y^3 + g_2y^2z + g_3yz^2 + g_4z^3)x^3$
$+ +(g_5y^4 + g_6y^3z + g_7y^2z^2 + g_8yz^3 + g_9z^4)x^2$
$+ (g_{10}y^5 + g_{11}y^4z + g_{12}y^3z^2 + g_{13}y^2z^3 + g_{14}yz^4 + g_{15}z^5)x$
$+ g_{16}y^6 + g_{17}y^5z + g_{18}y^4z^2 + g_{19}y^3z^3 + g_{20}y^2z^4 + g_{21}yz^5 + g_{22}z^6$
where g_i is polynomial of α, β and a_i ($1 \leq i \leq 20$).

If $(g_{15}, g_{21}, g_{22}) = (0, 0, 0)$, then the curve defined by the defining equation has a singular point at [0,0,1] in $\mathbb{P}^2$.

We consider the existence of solution for the system of algebraic equations $(g_{15}, g_{21}, g_{22}) = (0, 0, 0)$. Here, we use elimination theorem(Theorem2.1).

Let I be the polynomial ideal $\langle g_{15}, g_{21}, g_{22} \rangle$ and let G be a Gröbner basis of I with respect to lexicographic order where $\alpha > \beta > a_1 > \ldots > a_{20}$. Then, the curve defined by the defining equation of result 3.1 has a singular point at [0,0,1] if and only if the Gröbner base of $G \cap \mathbb{C}[a_1, \ldots, a_{20}]$ is equal to 0.

We take their 2-th elimination ideals as follows:

$$G_2 = G \cap \mathbb{C}[a_1, \ldots, a_{20}].$$

Here, G_2 is a Gröbner basis of the 2-th elimination ideal I_2. Then G_2 consists of polynomials of $a_1, \ldots, a_{20}$. For any $g_i \in G_2$, $g_i = 0$ is a necessary condition to get the restriction of the defining equation that the dimension of the parameter is equal to 19.

And we must show the existence of the solution (α, β) in the system of algebraic equations $(g_{15}, g_{21}, g_{22}) = (0, 0, 0)$ for any value of the parameter coefficients.

If this Gröbner basis G_2 is obtained, we can get the restriction of the defining equation that the dimension of the parameter is equal to 19. However, we cannot carry out this calculation at large-scale algebra calculation.

This Gröbner basis G_2 may become the restriction of the defining equation that the dimension of the parameter is equal to 19. However, a concrete calculation is not yet possible. If the parameters are concrete numerical values, it can calculate. But the case except it is not possible.

So, we tried also the following methods. We use a resultant ([6]).

Let $R(g_{15}, g_{21})$ be the resultant of g_{15} and g_{21} in α. Similarly, let $R(g_{15}, g_{22})$ be the resultant of g_{15} and g_{22} in α.

$$R(g_{15}, g_{21}) = \sum_{i=0}^{23} c_i \beta^i, \quad R(g_{15}, g_{22}) = \sum_{i=0}^{27} c'_i \beta^i$$

where c_i, c'_i are polynomials of a_i ($1 \leq i \leq 20$).

Here, we put $h := 81R(g_{15}, g_{22}) - R(g_{15}, g_{21})\beta^4$. Then $h = \sum_{i=0}^{26} c''_i \beta^i$.

We consider the resultant of $R(g_{15}, g_{21})$ and h in β.

This resultant $\neq 0$ may become the necessary condition of the restriction of the defining equation that the dimension of the parameter is equal to 19. However, this method also does not enable the final calculation either.

References

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