Weierstrass Theory of Abelian Integrals and its Realization in Sage

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It is well-known, that abelian integral is an integral of the form

$$\int R(x,y)dx,$$

here R is an arbitrary rational function of the two variables x and y related by the equation

$$f(x,y) = 0,$$

where f is an irreducible polynomial from $\mathbb{Q}[x,y]$. Mathematicians of 19th century considered the theory of abelian integrals as the necessary completion of mathematical analysis, many outstanding mathematicians were engaged in development of this theory and its application to differential equations, but after WWI works in this theory have died away. Now in modern CAS there are few packages for work with abelian integrals, probably, `Algcurves` for Maple (M. van Hoeij et all.) is the best of them. This package can calculate, e. g., the genus of a given curve or a basis for the linear space of differentials of the first kind, but can't decompose given integral to integrals of three kinds. Furthermore, for Maple this is a difficult problem even in the case of elliptic curves.

It should be noted that modern algorithms have been created without regard to Weierstrass works [1]. Probably, the matter is that Weierstrass didn't publish his results after 1870 [2]. His lectures on abelian integrals (1875) were published only in 1902 by G. Hettner and J. Knoblauch, so authors of well-known reviews on the theory of abelian integrals (Backer, 1897; Tikhomandritski, 1895) presented Weierstrass's ideas, relying on incomplete students manuscripts. In the 20th century there was a prejudice that Weierstrass theory is not constructive as opposed to Kronecker theory. In our opinion both theories are constructive, but work with different fields, with $\overline{\mathbb{Q}}$ and $\mathbb{Q}$ respectively.

Central notion of Weierstrass theory is a fundamental function for a given curve $f(x,y) = 0$. Let H be an element of the field $\mathbb{C}(x,y)$ and equation

$$H(x,y) = t$$

with fixed $t \in \mathbb{P}$ has r roots on the curve, then r is called an order (Grad) of the function H. If (x', y') is a point on this curve then there is such function $H \in \mathbb{C}(x, y)$ that (x', y') is a simple pole of H and the residue at the point is equal 1. Such function with minimal order $r = 1 + p$ is called a fundamental function (Hauptfunktion) and the number p is called a genus (Rang) of curve. In many cases we can write expression for fundamental function explicitly, so for elliptic curve

$$y^2 = a_0y^3 + a_1y^2 + a_2y + a_3$$

fundamental function is equal

$$\frac{1}{2y'}\frac{y + y'}{x - x'},$$

this case has been particularly studied by P.M. Pokrovski [3]. In Weierstrass lectures there is an algorithm for finding the fundamental functions based on power series expansion in local uniformization of a given curve.

Trivial statement of the existence of the fundamental function is the unique existence theorem in Weierstass's lectures, any other objects can be interpreted as derivatives of the fundamental function. By analogy with Green's function, Weierstrass considers not $H(x, y)$, but

$$H(x, y; x', y')dx'.$$

This dual construction is a rational function with respect to (x, y) and a differential with respect to (x', y'). Remarkably it turns out that $H(x, y; x', y')dx'$ with respect to second argument is an integral of the 3rd kind (Art) with the pole at (x, y). If $(a_1, b_1), \dots, (a_p, b_p)$ are poles of the fundamental function with respect to first argument and power series

$$x = \mathfrak{P}_x(t), \quad y = \mathfrak{P}_y(t)$$

give a local uniformization of a curve in the neighborhood of the point (a_n, b_n), then

$$H(\mathfrak{P}_x(t), \mathfrak{P}_y(t); x', y')dx' = H_n(x', y')dx' \cdot \frac{1}{t} + \dots - H_n'(x', y')dx' \cdot t + \dots$$

Coefficients of these expressions give us well-known abelian integrals of the 1st and the 2nd kinds. Green's function is symmetric and by analogy Weierstrass proves that

$$\begin{aligned}&\frac{d}{dx_1}H(x_1, y_1; x_2, y_2) - \frac{d}{dx_2}H(x_2, y_2; x_1, y_1) = \\ &= \sum_{n=1}^{p} H_n(x_2, y_2)H_n'(x_1, y_1) - H_n(x_1, y_1)H_n'(x_2, y_2),\end{aligned}$$

This fundamental equation plays the same role in Weierstrass theory that reduction formulas in the theory of rational integrals. So for any rational function R we can write abelian integral

$$\int R(x, y)dx$$

as sum of algebraic part $R'(x,y)$, log-part

$$\sum_m c_m H(x_m, y_m; x, y)dx$$

with log-singularities in poles of R and the 3rd part

$$\sum_{n=1}^{p} g_n' \int H_n(x,y)dx - g_n \int H_n'(x,y)dx$$

with simple poles in fixed singularities $(a_1, b_1), \dots, (a_p, b_p)$ of the fundamental function. In lectures by Weierstrass there are explicit formulas for calculation c_m, g_n, g_n' using the power series of local uniformization. This 3nd part is an elementary function iff all g_n and g_n' are equal to zero. So we have:

- decomposition of given abelian integral into integrals of three kinds,
- conditions for integrating given algebraic function in elementary functions, and
- equivalence between Weierstrass definition of genus and commonly used definition as dimension of linear space of homomorphic abelian integrals.

From Weierstrass definition follows that curves with $p = 0$ are biracionally equivalent to the projective line and curves with $p = 1$ are biracionally equivalent to the elliptic curve. Furthermore, we can explicitly write these transformations if we know the fundamental function.

Unfortunately in Weierstrass's lectures there are nocalculation examples, and we want to fill this gap. As it has been noted above the unique instrument of Weierstrass theory is decomposition in Puiseux series over $\mathbb{Q}$. Now in Sage there is a realization of $\overline{\mathbb{Q}}$ (`QQbar`) and polynomial rings over this field written by Carl Witty in 2007. In our talk we want to present some realization of Weierstrass theory in Sage. Russian retelling of Weierstrass's lectures can be found on our site [4].

References

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