On computer modeling of finite-generated free projective planes

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Abstract. This paper is devoted to computer modeling of the process of constructing free projective planes - more precisely, to algorithmically finding their successive incidence matrices - and also to considering some numerical characteristics of these matrices.

Introduction

Free projective planes were first introduced by M. Hall in his fundamental paper [1] where he considered their basic properties. Since then, these planes have become the subject of constant interest of mathematicians studying abstract algebraic structures, group theory and their representations, and so on [2, 3, 5, 6, 7]. There are also good surveys which one can use to get acquainted with the basic concepts and achievements of the modern theory of combinatorial geometries, for example, [4, 8].

This paper is devoted to computer modeling of the process of constructing free projective planes - more precisely, to algorithmically finding their successive incidence matrices - and also to considering some numerical characteristics of these matrices.

Remarks about notations: If A is a (non-empty) matrix then $dim1(A)$ (resp. $dim2(A)$) is a number of its rows (resp. columns); $[A]_{i,j}$ means its element at the entry (i, j); A_i (resp. A^j) means $i-$th row (resp. j-th column); $diag(A)$ for a square matrix A means column-vector of its diagonal elements; $Total[A]$ is a sum of all elements in A. Moreover, we treat binomial coefficient $\binom{x}{2}$ and differential operators (derivatives, Laplace operator) as listable functions.

$\eta_{i,j}$ is a column-vector with "1"$-$s only in two different positions i and j and all the rest components equal to "0"$-$s.

As a rule we do not show the matrix format explicitly until it is not clear from context.

E denotes identity matrix; J is a square constant matrix of (only) "1"$-$s; $J^* = J - E$; $\{\}$ denotes empty matrix; $\langle\ ,\ \rangle$ means Euclidean scalar product; for a matrix A and real α we define a product $\alpha \bullet A$ as follows: $\alpha \bullet A = \begin{cases} \alpha A, & \text{if } \alpha \neq 0 \\ \{\}, & \text{if } \alpha = 0. \end{cases}$

$A \circ B$ denotes the element-wise (Hadamard) product of matrices with the identical formats.

If A and B are matrices having appropriate formats then $A|\cup B$(resp. $\underline{A}\cup B$) denotes a concatenation of A and B from the right (resp. "from below") providing $A| \cup \{\} = \underline{A} \cup \{\} = A$.

1. Preliminaries

In this section we mostly follow the terminology and definitions of book [4].
Definition 1. A configuration (or a partial plane [1]) is a pair $\Pi = (P, L)$ where P is (nonempty) set of points and L is a family of subsets of P called lines under the condition that the following axiom is valid:

C1: Any two different points are incident with no more than one line.

Axiom C1 implies

C2: Any two different lines are incident with no more than one point in common. [4]

As a rule in this paper we shall be interested only in the case of finite sets P.
Examples 1.

1. Desargues and Pappus configurations are well-known (cf. [4]) .

2. If in Definition 1 $L = \emptyset$ and $|P| = m$, $m > 0$ is an integer, then we have a pure $m-$points configuration.

3. If L consists of all pairs $\{a, b\}$, $a, b \in P$, $a \neq b$ then $\Pi = (P, L)$ is a full graph on m vertices.

4. Let $\Pi^m = (P, L)$, $m \geq 4$ be a configuration with $|P| = m$ and only one line λ, (i.e. $L = \{\lambda\}$) where $|\lambda| = m - 2$. This means that all points besides two of them lie on the (unique) line λ. These configurations are called standard [8] or Hall configurations and were first introduced by M. Hall in his fundamental paper [1], p. 237.

Definition 2. Configuration $\Pi = (P, L)$ is called a projective plane, if in axioms C1 and C2 the words "...with no more than one..." are changed by "... exactly one...", i.e. in $\Pi = (P, L)$ the following axioms are valid:

P1: Any two different points are incident to exactly one line;

P2: Any two different lines are incident to exactly one point in common;

and in addition the axiom

P3: There exist 4 different points such that no three of them are collinear; in order to exclude some degenerate configurations (cf. [8]).

The following simple statements can be easily proved for a finite projective plane [2]:

A) Every line is incident to exactly $n + 1$ points;

B) Every point is incident to exactly $n+1$ lines;
C) $|P| = |L| = N = n^2 + n + 1$.
The number n is called the order of the finite projective plane.

"Prime-power hypothesis for the orders of the finite projective planes" claims that always $n = p^\mu$ for some prime p. Nowadays this hypothesis remains unproved.

If $\Pi = (P, L)$ is a finite configuration with $|P| = m$ and $|L| = l$, $l > 0$ then the incident matrix of Π is defined as $l \times m$ 0-1-matrix $A = (a_{i,j})$ where

$$a_{i,j} = \begin{cases} 1, & \text{if point } j \text{ is incident with line } i \\ 0, & \text{if point } j \text{ and the line } i \text{ are not incident} \end{cases} \quad 1 \le i \le L.1 \le j \le m \tag{1}$$

in some chosen (and fixed) numerations of sets P and L.

Obviously

$$\begin{aligned} Total[A_i] &= \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{i,j}^2 \\ &= \langle A_i, A_i \rangle = (\text{number of points on the } i - \text{th line}) \end{aligned} \tag{2}$$

$$\begin{aligned} Total[A^j] &= \sum_{i=1}^{l} a_{i,j} = \sum_{i=1}^{l} a_{i,j}^2 \\ &= \langle A^j, A^j \rangle = (\text{number of lines incedent to the } j - \text{th point}) \end{aligned} \tag{3}$$

$$Tr(AA^T) = Tr(A^T A) = Total(A) \tag{4}$$

If all the outside-diagonal elements in AA^T (resp., $A^T A$) are equal to 1, we say that configuration is line-wise ample (resp. point-wise ample).

Clearly, if $\Pi = (P, L)$ is a projective plane of order n then it is both point-wise ample and line-wise ample and its incident matrix is a square $N \times N$ 0-1-matrix such that

$$AA^T = A^T A = nE + J \tag{5}$$

(cf. for example, [2]).

2. Free projective plane generated by configuration

Let $\Pi_0 = (P_0, L_0)$ be some (initial) configuration. The free projective plane generated by Π_0 is defined by the following process:
1. Let $\Pi_1 = (P_1, L_1)$ be a new configuration where $L_1 = L_0$ and $P_1 = P_0 \cup \nu P_0$

$$\nu P_0 = \{(a)(b) | a, b \in L_0, a \text{ and } b \text{ are not incident in } \Pi_0\} \tag{6}$$

i.e. every pair of non-incident lines defines a new point named $(a)(b)$ which is "intersection" of lines a and b. Evidently Π_1 is line-wise ample.

2. Let $\Pi_2 = (P_2, L_2)$ be a new configuration where $P_2 = P_1$ and $L_2 = L_1 \cup \nu L_1$

$$\nu L_1 = \{(a)(b) | a, b \in P_1, a \text{ and } b \text{ are not incident in } \Pi_1\} \tag{7}$$

i.e. every pair of non-incident points a and b defines a new line named $(a)(b)$ which "connects" points a and b. Evidently Π_2 is point-wise ample.

Iterating this construction we get a sequence (finite or infinite) of configurations $\{\Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5, \ldots, \Pi_r, \ldots\}$ in which for r even we add new points to Π_r, as in item 1 and for r odd we add new lines to Π_r as in item 2 and get next configuration Π_{r+1}, $r \geq 0$.

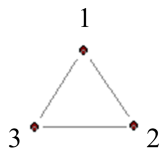
Proposition 1 (see [4]). If Π_0 contains 4 different points no three of which are collinear then $\Pi = fr(\Pi_0) = \bigcup_{k=0}^{\infty} \Pi_k$ is a projective plane.

This plane is said to be the free projective plane generated by Π_0.

Example 2

1. If Π_0 is a projective plane then evidently $fr(\Pi_0) = \Pi_0$.

2. If $|\Pi_0| = 3$ and $|L_0| = 0$ then $fr(\Pi_0)$ is called a "projective plane of order $n = 1$" (see Definition 2, p.1) and it is a plane over the field of one element (Fig.1, left). Its incident matrix is cyclic.



$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

FIGURE 1. Projective plane of order $n = 1$ (left) and its incident matrix (right).

The following theorem of M. Hall (see [1]) explains the importance of Hall configuration Π^4 :

1) Let Π_0 is any non-degenerate configuration but not a projective plane. Then $fr(\Pi_0)$ contains $fr(\Pi^4)$ as a subplane. Moreover, such plane is never desarguesian.

2) A $fr(\Pi^m)$, $m \geq 4$ contains $fr(\Pi^{m+1})$.

Everywhere in what follows we deal only with the Hall configuration Π^4, i.e. $fr(\Pi^4) = \{\Pi^4_r\}_{r=0,1,2,\ldots}$, that is "free equivalent"(see [1]) to pure configuration on 4 points, or full graph with 4 vertices.

3. Matrix approach

According to what was said at the end of previous section we begin with configuration $\Pi_0 = \Pi^4$ (which is zero-step, $s = 0$, of our algorithm) with incident matrix

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

which corresponds to the configuration 3 from Example 1 with $m = 4$. This configuration (tetrahedron) is shown below on Fig. 2 (left), where the numeration of lines is omitted.

Evidently here $\dim 1(A_0) = \Lambda_0 = 6$, $\dim 2(A_0) = P_0 = 4$. Since

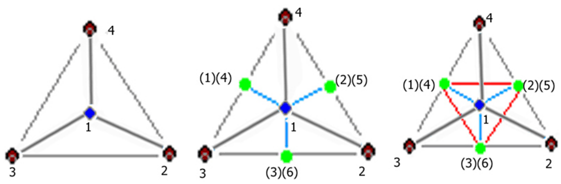


FIGURE 2. Initial configuration $\Pi_0 = \Pi^4$ (left) and two steps of the algorithm: adding new points (center) and new lines (right).

$$A_0 A_0^T = \begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \end{pmatrix} \qquad A_0^T A_0 = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

this configuration is point-wise ample (any two different points are incident), but is not line-wise ample because exactly 3 pairs of lines, namely 1,4, 2,5 and 3,6, have no points in common.

According to item 1 of the general constructing of $fr(\Pi_0)$ at the next step $s = 1$ we must add to Π_0 $\nu P_0 = 3$ new points, namely (1)(4), (2)(5) and (3)(6) (see Fig. 2), that means that we must concatenate (from the right) to A_0 three new columns numbered respectively 5, 6, 7, whereas the amount of new lines $\nu A_0 = 0$.

So, here $\dim 1(A_1) = \Lambda_1 = \Lambda_0 = 6$, $\dim 2(A_1) = P_0 + \nu P_0 = 4 + 3 = 7$ and the matrix of the next configuration Π_1 (see Fig.2 (center)) is

$$A_1 = \left(\begin{array}{cccc|ccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Note that positions of "1"−s in the concatenated columns are (clearly why) exactly 1 and 4, 2 and 5, and 3 and 6.

Going over to the next step $s = 2$ we find that

$$A_1A_1^T = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix} \quad A_1^TA_1 = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2 \end{pmatrix}$$

So, here $\dim 1(A_2) = \Lambda + 2 = \Lambda_1 + \nu\Lambda_1 = 6 + 3 = 9$, $\dim 2(A_2) = P_2 = P_1 + \nu P_1 = 7 + 0 = 7$ and the matrix of the next configuration (see Fig.2 center) is

$$A_2 = \left[\begin{array}{ccccccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Now it is not difficult to describe the general case for any step $s > 0$:

a1) If $s \equiv 1 \mod 2$ we add new points

$$\begin{aligned} \nu P_{s-1} &= \text{(number of non-incident lines at step } s-1) \\ &= \frac{1}{2}\,\text{(number of "0"-s in } A_{s-1}A_{s-1}^T) \\ &= \binom{\Lambda_{s-1}}{2} - Total\left[\binom{diag(A_{s-1}^TA_{s-1})}{2}\right] \\ &= \binom{\Lambda_{s-1}}{2} - Total\left[\binom{A_{s-1}^TA_{s-1}}{2}\right] \end{aligned} \tag{8}$$

whereas clearly $\nu A_{s-1} = 0$.

So, in this case we get a formula (we remind that $0 \bullet a = \{\}$):

$$A_s = A_{s-1}| \bigcup_{2\le i\le A_{s-1} i\le j\le A_{s-1}} \big((1 - [A_{s-1}A_{s-1}^T]_{i,j}) \bullet \eta_{i,j}\big) \tag{9}$$

Dually,

a2) If $s \equiv 0 \mod 2$ we add new lines

$$\begin{aligned}
\nu\Lambda_{s-1} &= \text{(number of non-incident points at step } s-1 \\
&= \frac{1}{2}\,\text{(number of "0"-s in } A_{s-1}^T A_{s-1}) \\
&= \binom{P_{s-1}}{2} - Total\left[\binom{diag(A_{s-1}A_{s-1}^T)}{2}\right] \\
&= \binom{P_{s-1}}{2} - Total\left[\binom{A_{s-1}A_{s-1}^T}{2}\right]
\end{aligned} \tag{10}$$

whereas clearly $\nu P_{s-1} = 0$.

For example, for $s = 2$ we get $\nu\Lambda_1 = \binom{7}{2} - 6 \cdot \binom{3}{2}$, since $P_1 = 7$, $diag(A_1 A_1^T) = (3\,3\,3\,3\,3\,3)$. So, in this case we get a formula:

$$A_s = \underline{A_{s-1}} \bigcup_{2 \le i \le A_{s-1} i \le j \le A_{s-1}} \left((1 - [A_{s-1}^T A_{s-1}]_{i,j}) \bullet \eta_{i,j}^T\right) \tag{11}$$

Formulas (9) and (11) give rise to the first variant of our algorithms.

4. Bilinear forms approach

Let $\pi = \{p_i\}_{i=1}^{\infty}$ and $\lambda = \{l_i\}_{i=1}^{\infty}$ be two sets of independent variables for points and lines respectively.

For any step $s \ge 0$ we introduce a bilinear form $F_s = F_s(\pi, \lambda) = \pi^T A_s \lambda$ where A_s is an incident matrix constructed on step s (see Sec. 3) and π and λ are initial segments of the infinite sequences of variables π and λ having appropriate lengths. For example, for $s = 0$ we have $\pi = \{p_i\}_{i=1}^4$, $\lambda = \{l_i\}_{i=1}^6$ and

$$\begin{aligned}
F_0 &= l_4p_1 + l_5p_1 + l_6p_1 + l_2p_2 + l_3p_2 + l_4p_2 + l_1p_3 + l_3p_3 + l_5p_3 + l_1p_4 \\
&\quad + l_2p_4 + i_6p_4 \\
&= l_1(p_3 + p_4) + l_2(p_2 + p_4) + l_3(p_2 + p_3) + l_4(p_1 + p_2) + l_5(p_1 + p_3) \\
&\quad + l_6(p_1 + p_4) \\
&= p_1(l_4 + l_5 + l_6) + p_2(l_2 + l_3 + l_4) + p_3(l_1 + l_3 + l_5) + p_4(l_1 + l_2 + l_6)
\end{aligned} \tag{12}$$

Now it is clear that also in general case $Coefficient[F_s, l_i] = \frac{\partial F_s}{\partial l_i}$ is a linear form in π representing the $i-$th row of A_s; $Coefficient[F_s, p_j] = \frac{\partial F_s}{\partial p_j}$ is a linear form in λ representing the $j-$th column of A_s.

Also it is clear that two lines, l_i and l_k with $1 \le i, k \le \Lambda_s$, $i \ne k$ are not incident iff. the linear forms $\frac{\partial F_s}{\partial l_i}$ and $\frac{\partial F_s}{\partial l_k}$ have no variables in common that implies that in this case the Laplace operator in π

$$\Delta_\pi \left(\frac{\partial F_s}{\partial l_i} \cdot \frac{\partial F_s}{\partial l_k}\right) = \sum_{p \in \pi} \frac{\partial^2}{\partial p^2} \left(\frac{\partial F_s}{\partial l_i} \cdot \frac{\partial F_s}{\partial l_k}\right) = 0 \tag{13}$$

and otherwise

$$\Delta_\pi\left(\frac{\partial F_s}{\partial l_i}\cdot\frac{\partial F_s}{\partial l_k}\right)=2. \tag{14}$$

It's clear that if $i=k$ then

$$\lambda_\pi\left(\left(\frac{\partial F_s}{\partial l_i}\right)^2\right)=2\cdot\ \text{(number of points on } i-\text{th line)}=2\left[diag(A_sA_s^T)\right]_i \tag{15}$$

For example,

$$\Delta_\pi\left(\frac{\partial F_0}{\partial l_1}\cdot\frac{\partial F_0}{\partial l_4}\right)=\sum_{r=1}^{4}\frac{\partial^2}{\partial p_r^2}\left((p_3+p_4)(p_1+p_2)\right)=0,$$

whereas

$$\Delta_\pi\left(\frac{\partial F_0}{\partial l_1}\cdot\frac{\partial F_0}{\partial l_2}\right)=\sum_{r=1}^{4}\frac{\partial^2}{\partial p_r^2}\left((p_3+p_4)(p_2+p_4)\right)=2,$$

and

$$\Delta_\pi\left(\frac{\partial F_0}{\partial l_1}\right)^2=\sum_{r=1}^{4}\frac{\partial^2}{\partial p_r^2}\left((p_3+p_4)^2\right)=2\cdot 2=4.$$

Obviously that formulas dual to (13), (14) and (15) also are valid mutatis mutandis.

Using formulas (13), (14), (15) and their duals it is easy to verify matrices equalities

$$\frac{1}{2}\Delta_\pi\left(\left(\frac{\partial F_s}{\partial\lambda}\right)^{\otimes 2}\right)=A_sA_s^T,\ \frac{1}{2}\Delta_\lambda\left(\left(\frac{\partial F_s}{\partial\pi}\right)^{\otimes 2}\right)=A_s^TA_s, \tag{16}$$

where $\frac{\partial F_s}{\partial\lambda}=grad_\lambda(f_s)$, $\frac{\partial F_s}{\partial\pi}=grad_\pi(f_s)$, the Laplace operators are supposed to be listable and $^{\otimes 2}$ means tensor square.

These formulas also give rise to alternative algorithm for recursive construction of $fr(\Pi^4)$.

5. Implementation

As was said above we used "matrix approach", and "bilinear forms approach".

The first difficulty in programming was caused by the requirement to avoid zero-columns/rows in incident matrices as well as "fictitious" variables in bilinear forms. This difficulty is surmounted with special procedures for numeration of new constructed columns/rows of matrices and new variables of bilinear forms.

Rather more serious obstacle is the (mentioned above) fact of the very fast growth of matrices' formats. Though those are very sparse 0-1-matrices, the programming tools for such matrices provided by Mathematica occurred to be not sufficient for our purposes, so, the computer memory resources became exhausted very soon...

So, we managed to calculate only 7 members of the sequence $u_n=\nu P_n+\nu\Lambda_n$, $n\geq 0$ (note that one of the two summand in "u_n" is always equal to 0):

3, 3, 6, 24, 282, 37233, 684792168, ...

It is easy to check empirically that this sequence grows asymptotically as a double exponent of n (Fig. 3).

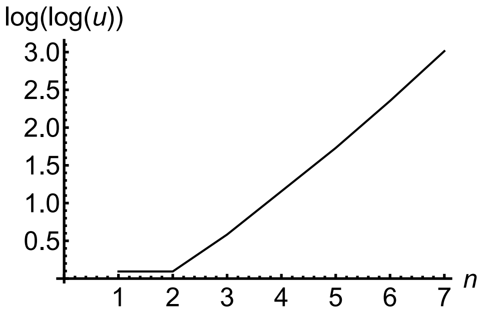


FIGURE 3. Number of elements grows as double exponent (linear on log(log) scale.

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