On Matveev's question about virtual 3-manifolds

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Abstract. S. V. Matveev introduced the concept of virtual 3-manifolds and posed a problem, whether the natural map from the set of nondegenerate virtual 3-manifolds to the set of 3-manifolds with RP^2-singularities is injective. We show that this map is not injective and present an infinite family of virtual 3-manifolds corresponding to the same 3-manifold with RP^2-singularities.

Matveev's virtual 3-manifolds can be naturally defined either in terms of *spines* or in the 'dual' language of singular triangulations. Contrary to the original definition via spines, we start with triangulations.

Face identification schemes. Let n be a positive integer, let $\mathcal{D} = \{\Delta_1, \dots, \Delta_n\}$ be a set of n disjoint tetrahedra, and let $\Phi = (\phi_1, \dots, \phi_{2n})$ be a collection of $2n$ affine homeomorphisms between the facets (i. e., triangular faces) of tetrahedra in $\mathcal{D}$ such that each facet has a unique counterpart. Following [Mat03, p. 11], we refer to the pair $(\mathcal{D}, \Phi)$ as to a *face identification scheme* (a *scheme*). The *quotient space* $Q := Q(\mathcal{D}, \Phi)$ of the scheme $(\mathcal{D}, \Phi)$ is defined as the space obtained from $\mathcal{D}$ by identification of faces via the homeomorphisms in Φ. For any scheme, Q is either a genuine or a singular 3-manifold (see [Sei33] or [Mat03, Proposition 1.1.23]). All singular points of Q correspond either to vertices or to barycenters of edges of the tetrahedra in $\mathcal{D}$. The latter happens if the quotient map folds an edge so that symmetric points (with respect to the barycenter of the edge) have the same image. If a singularity point x in Q corresponds to the barycenter of an edge, then the link of x is himeomorphic to the *projective plane* RP^2, i. e., x is an RP^2-singularity.

Pseudo-Pachner move. In Euclidean three-space, let P be the convex hull of five points that are in general position. Assume that P is not a tetrahedron. Then P is the union of two geometric tetrahedra (Δ_1 and Δ_2, say) and at the same time P is the union of three geometric tetrahedra with disjoint interiors (Δ_3, Δ_4, and Δ_5, say). Let

$$\tau = \partial\Delta_1 \cup \partial\Delta_2 \qquad \text{and} \qquad \tau' = \partial\Delta_3 \cup \partial\Delta_4 \cup \partial\Delta_5.$$

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If $Q := Q(\mathcal{D}, \Phi)$ is the quotient space of the scheme $(\mathcal{D}, \Phi)$, we denote by $\sigma(Q)$ the image in Q of boundaries of tetrahedra in $\mathcal{D}$ (*2-skeleton* of $(\mathcal{D}, \Phi)$).

We say that two schemes $(\mathcal{D}, \Phi)$ and $(\mathcal{D}', \Phi')$ are *related by a pseudo-Pachner move* if there exists a homeomorphism $h\colon Q \to Q'$ between the quotient spaces $Q := Q(\mathcal{D}, \Phi)$ and $Q' := Q(\mathcal{D}', \Phi')$ and a map $f\colon P \to Q'$ such that

– the restriction of f to the interior of P is an embedding;

– $h(\sigma(Q)) \cap f(P) = f(\tau)$ and $\sigma(Q') \cap f(P) = f(\tau')$;

– $h(\sigma(Q)) \setminus f(P) = \sigma(Q') \setminus f(P)$, that is, $h(\sigma(Q))$ and $\sigma(Q')$ do coincide outside $f(P)$.

Definition. Virtual 3-manifolds. We say that two face identification schemes are *equivalent* if they are related by a chain of pseudo-Pachner moves. A *virtual* 3-*manifold* is an equivalence class of schemes. We say that a virtual 3-manifold is *degenerate* if it corresponds to a scheme with precisely one tetrahedron.

Truncated quotient spaces as underlying spaces of virtual 3-manifolds. The *truncated quotient space* $Q_t := Q_t(\mathcal{D}, \Phi)$ of the scheme $(\mathcal{D}, \Phi)$ is defined as the space obtained from the quotient space $Q := Q(\mathcal{D}, \Phi)$ by deleting small open proper neighborhoods of points in Q that are images of vertices of the tetrahedra in $\mathcal{D}$. Truncated quotient spaces of equivalent schemes are obviously homeomorphic. By the *underlying space* $Q_t(V)$ of a virtual 3-manifold V we will mean the truncated quotient space of schemes in V. Thus, the underlying space of a virtual 3-manifold is a 3-manifold with RP^2-singularities.[1] The following facts follow from key results of the *special spine theory* (see [Mat03, Mat09]).

Corollary 1. (1) *Each compact connected* 3*-manifold M with RP^2-singularities and nonempty boundary is the underlying space of a virtual* 3*-manifold.*

(2) *If M is, moreover, genuine, then it is the underlying space of a unique nondegenerate virtual* 3*-manifold.*

Problem. The following natural question arises (see [Mat09]). Do the second statements of Corollary 1 holds for 3-manifolds with RP^2-singularities? That is, are there distinct nondegenerate virtual 3-manifolds with the same connected underlying space?

We present the following theorem that answers the question above.

Theorem 1. *The cone* $\mathrm{C}(RP^2) = (RP^2 \times [0, 1])/(RP^2 \times \{0\})$ *over the projective plane RP^2 is the underlying space for an infinite number of pairwise distinct virtual* 3*-manifolds.*

In order to explain the idea of the proof of Theorem 1, we introduce the concepts of 3*-manifolds with traced RP^2-singularities.*

[1]A compact three-dimensional polyhedron W is called a 3*-manifold with RP^2-singularities* if the link of any point of W is either a 2-sphere, or a 2-disc, or RP^2.

3-manifolds with traced RP^2-singularities. A 3-*manifold with traced RP^2-singularities* is a pair $(W, \mathcal{I})$, where W is a 3-manifold with RP^2-singularities and $\mathcal{I}$ is a subpolyhedron of W with the following properties:

– $\mathcal{I}$ is a disjoint union of arcs (these arcs are called *traces*),

– the number of components in $\mathcal{I}$ equals the number of singular points in W,

– each component of $\mathcal{I}$ emanates from the boundary of W and ends at a singularity point of W.

We describe two interpretations for 3-manifolds with traced RP^2-singularities.

3-manifolds with Möbius singularities. We say that a compact 3-dimensional polyhedron W is a 3-*manifold with Möbius singularities* if the link of any point of W is either a 2-sphere, or a 2-disc, or a Möbius band. (The set of points of W whose links are 2-discs or Möbius bands form the *boundary* ∂W of W, so that the singularities of W are contained in its boundary.) Collapsing traces to points, we see that the set of 3-manifold with traced RP^2-singularities is in a natural one-to-one correspondence with the set of 3-manifolds with Möbius singularities.

3-manifolds decorated with orientation-reversing curves. A closed simple curve γ on a surface is *orientation-reversing* if a tubular neighbourhood of γ is a Möbius band. If W is a genuine 3-manifold and C is a collection of closed simple pairwise disjoint orientation-reversing curves on the boundary ∂W, then collapsing each curve in C to a point transforms W to a 3-manifolds with Möbius singularities (each curve of C gives a singularity point). Thus, the set of 3-manifolds equipped with orientation-reversing curves is another interpretation for the set of 3-manifolds with traced RP^2-singularities.

From virtual 3-manifolds to 3-manifolds with traced RP^2-singularities. Each virtual 3-manifold is naturally assigned with a 3-manifold with traced RP^2-singularities. Indeed, observe that all singular points of the truncated quotient space $Q_t := Q_t(\mathcal{D}, \Phi)$ of the scheme $(\mathcal{D}, \Phi)$ are RP^2-singular points corresponding to barycenters of edges of the tetrahedra of $\mathcal{D}$. The quotient map folds the edges that give singular points so that symmetric points (with respect to the barycenters) of each such *folding* edge have the same image. We observe that the pair $(Q_t, \mathcal{I})$, where $\mathcal{I}$ is the image in Q_t of all folding edges, is a 3-manifold with traced RP^2-singularities. The described correspondence $(\mathcal{D}, \Phi) \mapsto (Q_t, \mathcal{I})$ associates a 3-manifold with traced RP^2-singularities to each scheme. Now, observe that any pseudo-Pachner move changes neither Q_t nor the subset in Q_t formed by the images of folding edges, so that equivalent schemes yield the same 3-manifold with traced RP^2-singularities. This gives a map (which we denote by F) from the set of virtual 3-manifolds to the set of 3-manifolds with traced RP^2-singularities. Note that by construction, forgetting the traces in $F(V)$, where V is a virtual 3-manifold, converts $F(V)$ to the underlying space of V.

Theorem 2. *If $(M, \mathcal{I})$ is a compact connected 3-manifold with traced RP^2-singularities and nonempty boundary, then there exists a virtual 3-manifold V such that $F(V) = (M, \mathcal{I})$.*

Lemma 1. *The cone* $\mathrm{C}(RP^2) = (RP^2 \times [0,1])/(RP^2 \times \{0\})$ *over the projective plane* RP^2 *is the base space for an infinite number of pairwise distinct* 3*-manifolds with traced* RP^2*-singularities. In other words, the singularity point in* $\mathrm{C}(RP^2)$ *can be traced in an infinite number of pairwise nonequivalent ways.*

Ideas of proofs. Theorem 1 follows from Theorem 2 and Lemma 1.

Theorem 2 can be proved in terms of special spine theory on the base of results established in [Mat03]. Given a compact connected 3-manifold with traced RP^2-singularities $(M, \mathcal{I})$, we delete from M open proper neighbourhoods of $\mathcal{I}$. The obtained genuine 3-manifold $M_{\mathcal{I}} \subset M$ possesses a *special spine* S (see [Cas65] or [Mat03, Theorem 1.1.13]). Observe that $\partial M_{\mathcal{I}} \setminus \partial M$ is the union of disjoint open Möbius bands. Let $C \subset \partial M_{\mathcal{I}}$ be the union of center curves of these Möbius bands. A natural retraction $p \colon M_{\mathcal{I}} \to S$ sends C to a collection of curves on S. We attach to S new *unthickenable* 2-cells along $p(C)$ and show that the obtained special polyhedron represents a virtual 3-manifold V with $F(V) = (M, \mathcal{I})$.

In order to prove Lemma 1, consider 'knotted' traces and use the branched double covering of $\mathrm{C}(RP^2)$ by the 3-ball (ramified over the singularity point). $\square$

References

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