# Groups and Types

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We consider some groups that appear naturally in type theories.

In each type theory $T$ a natural notion of isomorphism of types is defined (*cf.* [Di Cosmo 1995]). For each type $A$ one may define the groupoid $Gr(A)$ whose objects are all types $A'$ in $T$ isomorphic to $A$. Morphisms of $Gr(A)$ are isomorphisms between such types.

For each $A$ the group $Aut(A)$ of automorphisms (i.e., isomorphisms $A \to A$) is also defined.

There are some other groups structures naturally associated with $A$. We consider: (i) the group of permutations $\Sigma(A)$ that respect the isomorphisms relation $\sim$: $\forall \sigma \in \Sigma(A).(\sigma(A) \sim A)$; (ii) the groupoid $Gr_\Sigma(A) \subseteq Gr(A)$ whose objects are $A' \sim A$ such that $\exists \sigma \in \Sigma(A).(A' = \sigma(A))$ and morphisms are the same isomorphisms as in $Gr(A)$ (*i.e.*, it is a full subcategory of $Gr(A)$; (iii) other groups generated by composition of isomorphisms and permutations that respect the isomorphism of types.

The purpose of our work is the study of the connections between these algebraic structures. We consider in detail the simply typed lambda-calculus $\lambda^1\beta\eta$ and the second order system $\lambda^2\beta\eta$ (see [Di Cosmo 1995]), but look to some extent at other systems, e.g., systems with depenent types, with coproduct etc.

**Theorem 1.** For every finite group $G$ there exists some type $A$ in $\lambda^1\beta\eta$ such that the group $\Sigma(A)$ is isomorphic to $G$.

**Theorem 2.** Let $A$ be some type in $\lambda^1\beta\eta$ and $\overline{\forall}.A$ its universal closure (the type in the second order calculus $\lambda^2\beta\eta$). Then the group of automorphisms $Aut(\overline{\forall}.A)$ (in $\lambda^2\beta\eta$) is isomorphic to the cartesian product $Aut(A) \times \Sigma(A)$.

From these two theorems we derive the corollary that:

**Corollary.** For every finite group $G$ there exists some type $A$ in $\lambda^1\beta\eta$ such that the group $Aut(\overline{\forall}.A)$ (in $\lambda^2\beta\eta$) is isomorphic to $G$.

## References

[Di Cosmo 1995] Di Cosmo, R. (1995) *Isomorphisms of types: from lambda-calculus to information retrieval and language design.* Birkhauser.

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