Estimates on the semi-meandric crossing number of classical knots

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Abstract. A plane diagram D of a knot is said to be semi-meandric if D is the union of two simple smooth arcs. Every tame knot has a semi-meandric diagram. We use this fact to define a new knot invariant: the semi-meandric crossing number. Applying the technique of Gauss Codes and a specific algorithm transforming arbitrary knot diagrams to semi-meandric ones we obtain estimates on this invariant.

Introduction

Definition 1. A smooth closed plane curve is called *semi-meandric* if it is the union of two simple smooth arcs (an arc is called *simple* if it is non-self-intersecting).

Theorem 1. *Every tame knot has a semi-meandric diagram.*

Proof. It is shown in [1] that a knot which is dyed with two different colours can be projected on a plane without crossing strands of the same colour. Clearly, such projection gives us a semi-meandric diagram. □

Remark 1. The approach developed in [1] is based on braid theory. We use a different approach, which gives another proof of Theorem 1.

Theorem 1 allows us to define a new knot invariant: the semi-meandric crossing number.

Definition 2. Recall that the *crossing number* of a knot K (denoted $\mathrm{Cr}(K)$) is the smallest number of crossings in any diagram of K. We define the *semi-meandric crossing number* of K (denoted $\mathrm{Cr_s}(K)$) as the smallest number of crossings in any semi-meandric diagram of K.

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Theorem 2. *For each tame knot K, the following inequalities hold*

$$\mathrm{Cr}(K) \leq \mathrm{Cr}_{\mathrm{s}}(K) < (\sqrt{3})^{\mathrm{Cr}(K)}.$$

Our proof of Theorem 2 is based on properties of Gauss Codes and on the algorithm presented below, which transforms arbitrary knot diagrams to semi-meandric ones.

Obtaining semi-meandric diagrams

Given a diagram D of a knot K, we obtain a semi-meandric diagram of K using the following algorithm.

1. We choose a simple arc J in D such that no endpoint of J is an intersection point of D (the interior of J is allowed to contain intersection points of D).
2. We choose an endpoint s of J and start walking along D from s until we find the first intersection point x of D that does not belong to J. (We may intersect J while traveling from s to x.) Denote the path connecting s and x by $[s, x]$.
3. We transform D by „pulling“ x along $[s, x]$. This transformation decreases the number of intersection points that are not in J, while new intersection points of D appear on J.
4. Until our diagram is not semi-meandric, repeat steps 2 and 3.

Observe that each time, when choosing an endpoint s of J at step 2, we have two possibilities. So, if D has n intersection points (before the first transformation) and J contains m of them, then we have 2^{n-m} possible final states of our diagram. If we use greedy algorithm and minimize the number of intersection points each time at step 2, then properties of Gauss Codes imply that double implementation of above procedure at most triples the number of intersection points on J. In other words, we prove that amongst four possible ways of decreasing by two the number of intersection points that are not in J, there exists at least one way where the number of intersection points of D in J at most triples.

Gauss Codes

Definition 3. The *Gauss Code* of a diagram D of a knot is obtained as follows:

- Label the intersection points of D with integers from 1 to n, where n is the number of this points.
- Start „walking“ along D, taking note of the labels of the intersection points we have gone through. If in a given intersection point we cross on the „over“ strand, write down the label of this point. Otherwise, we write down the negative of the label of the intersection point.

Remark 2. In terms of Gauss Codes, we can give an alternative definition of semi-meandric diagrams. A diagram with n intersection points is semi-meandric if and only if its cycled Gauss Code splits in two „connected“ parts such that the absolute values of elements in each part is precisely all numbers from 1 to n.

Remark 3. The algorithm from the previous section can be rewritten in terms of Gauss Codes. This allows us to make its exact implementation.

References

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[2] Ljiljana Radovic, Slavik Jablan, *Meander knots and links.* Filomat **29**:10 (2015), 2381–2392.

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