Factorization Operator Method for Solving BVP Exactly and Finding Eigenvalues and Eigenvectors

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Abstract. We consider a boundary value problem $\mathbf{B}_1 x = f$ where the linear operator $\mathbf{B}_1$ can be decomposed in the form $\mathbf{B}_1 = B_G^2 B_{G_0}^2$ with B_G and B_{G_0} being two linear operators of a special type. If the operator $\mathbf{B}_1$ is correct then the solution can be obtained in closed form. Moreover, the eigenvalues and the eigenvectors of the operator $\mathbf{B}_1$ are computed analytically. A partial integro-differential problem is solved to demonstrate the efficiency of the method.

1. Introduction

The study of many phenomena in science, engineering and economics involve advance mathematical models which in general have a high degree of complexity and they cannot be solved exactly. In these cases powerful numerical methods are usually employed to obtain the solution approximately. Some other problems can be transformed to simpler ones which it is easier to deal with and even to solve them explicitly, see for example [1] and [2] . The present article is a sequel of the work [3] by the same authors as above and discusses the exact solution of a boundary value problem involving an operator factored into two quadratic operators. In particular, we consider the boundary value problem $\mathbf{B}_1 x = f$ where the linear operator $\mathbf{B}_1$ has a decomposition of the form $\mathbf{B}_1 = B_G^2 B_{G_0}^2$ with B_G and B_{G_0} being two linear operators of a special type. We prove that if the operator $\mathbf{B}_1$ is correct then the solution can be obtained in closed form. Moreover, the eigenvalues and the eigenvectors of the operator $\mathbf{B}_1$ can be computed analytically.

We prove a theorem concerning the computation of the determinants of a special class of matrices and two corollaries regarding the evaluation of their eigenvalues and eigenvectors. Next we prove the main theorem for solving boundary value problems involving products of operators. An example problem with an integro-differential operator is chosen to test the efficiency of the method.

2. Special Matrices

Theorem 1. *Let the vectors* a, $b \in R^m$ *and the matrix*

$$C = I_m + b^t a.$$

where I_m *stands for the* $m \times m$ *identity matrix and* $b^t = col(b)$*. Then,*

$$|C| = \det(I_m + b^t a) = 1 + ab^t,$$

and for $|C| \neq 0$,

$$C^{-1} = \frac{1}{|C|}(|C|I_m - b^t a) = I_m - \frac{1}{|C|} b^t a.$$

□

From Theorem 1 some other results can be derived which are contained in the next two corollaries and are used in the present paper.

Corollary 1. *Let* $a, b \in R^m$*, the matrix* $C = I_m + b^t a$*, and* $|C| = \det C$*. Then the next statements are true:*
(i) The number $|C|$ *is an eigenvalue of the matrix* C *and* b^t *its corresponding eigenvector, namely*

$$Cb^t = (I_m + b^t a)b^t = |C|b^t.$$

(ii) If $|C| \neq 0$*, then the number* $\frac{1}{|C|}$ *is an eigenvalue of the matrix* C^{-1} *and* b^t *is its corresponding eigenvector, explicitly*

$$C^{-1}b^t = \frac{1}{|C|}\big(|C|I_m - b^t a\big)b^t = \frac{1}{|C|}b^t.$$

(iii) The number $|C|$ *is an eigenvalue of the matrix* C^t *and* a^t *is its corresponding eigenvector, that is to say*

$$aC = a|C| \quad or \quad C^t a^t = |C|a^t.$$

(iv) If $|C| \neq 0$*, then the number* $\frac{1}{|C|}$ *is an eigenvalue of the matrix* $(C^t)^{-1}$ *and* a^t *is its corresponding eigenvector, specifically*

$$aC^{-1} = \frac{1}{|C|}a \quad or \quad (C^t)^{-1}a^t = \frac{1}{|C|}a^t.$$

(v)If $|C| \neq 0$ *then,*

$$1 - aC^{-1}b^t = \frac{1}{|C|}.$$

□

Corollary 2. *Let the vectors* a, b, $c \in R^m$, $k \in \mathbb{C}$ *and the matrix*

$$C_1 = \begin{pmatrix} I_m + kb^t a & kb^t c \\ b^t a & I_m + b^t c \end{pmatrix} = I_{2m} + \begin{pmatrix} kb^t \\ b^t \end{pmatrix} \begin{pmatrix} a & c \end{pmatrix}.$$

Then, $\det C_1 = 1 + (ka + c)b^t$ *and for* $\det C_1 = |C_1| \neq 0$,

$$\begin{pmatrix} a & c \end{pmatrix} C_1^{-1} = \frac{1}{|C_1|} \begin{pmatrix} a & c \end{pmatrix} \quad \text{and} \quad 1 - \begin{pmatrix} a & c \end{pmatrix} C_1^{-1} \begin{pmatrix} kb^t \\ b^t \end{pmatrix} = \frac{1}{|C_1|},$$

$$C^{-1} = \frac{1}{|C_1|} \begin{pmatrix} |C_1| I_m - kb^t a & -kb^t c \\ -b^t a & |C_1| I_m - b^t c \end{pmatrix} = I_{2m} - \frac{1}{|C_1|} \begin{pmatrix} kb^t \\ b^t \end{pmatrix} \begin{pmatrix} a & c \end{pmatrix}.$$

□

3. Factorization Operator Method

We cite now the main theorem of the current work.

Theorem 2. *Let the operator* $\mathbf{B}_1 : H \to H$ *be defined by*

$$\begin{aligned} \mathbf{B}_1 x \quad = \quad & \widehat{A}^2 \widehat{A}_0^2 x - V \langle \widehat{A}_0 x, \Phi^t \rangle_{H^m} - Y \langle \widehat{A}_0^2 x, \Phi^t \rangle_{H^m} - S \langle \widehat{A} \widehat{A}_0^2 x, F^t \rangle_{H^m} \\ & - G \langle \widehat{A}^2 \widehat{A}_0^2 x, F^t \rangle_{H^m} = f, \quad D(\mathbf{B}_1) = D(\widehat{A}^2 \widehat{A}_0^2), \end{aligned} \tag{1}$$

where $\widehat{A}_0$, $\widehat{A} : H \to H$ *are linear correct operators and the vectors* $F, \Phi \in H^m$. *We also suppose that* x_0 *is an eigenvector of both operators* $\widehat{A}_0$ *and* $\widehat{A}$, *the numbers* α_0, α *are the corresponding to* x_0 *nonzero eigenvalues of the operators* $\widehat{A}_0$ *and* $\widehat{A}$, *respectively. Finally, let* $Y = (y_1 x_0, ..., y_m x_0) = y x_0$, $G = (g_1 x_0, ..., g_m x_0) = g x_0$ *and* $S, V \in H^m$, *where* $y_i, g_i \in \mathbb{C}$, $i = 1, ..., m$. *Then the following statements hold:*
(i) If

$$S = \alpha D g x_0, \quad V = \alpha_0 D_0 y x_0, \tag{2}$$

where $D_0 = 1 - \dfrac{1}{\alpha^2 D^2} y \langle x_0, \Phi^t \rangle_H$, *and* $D = 1 - g \langle x_0, F^t \rangle_H \neq 0$, *then,* $\mathbf{B}_1$ *has the unique decomposition*

$$\mathbf{B}_1 = B_G^2 B_{G_0}^2, \tag{3}$$

where

$$B_{G_0} x = \widehat{A}_0 x - G_0 \langle \widehat{A}_0 x, \Phi^t \rangle_{H^m} = f, \quad D(B_{G_0}) = D(\widehat{A}_0), \tag{4}$$

$$B_G x = \widehat{A} x - G \langle \widehat{A} x, F^t \rangle_{H^m} = f, \quad D(B_G) = D(\widehat{A}), \tag{5}$$

with $G_0 = \dfrac{1}{\alpha^2 D^2} y x_0$.
(ii) If G_0, S, V *satisfy (2) then* x_0 *is also the eigenvector of the operators* B_G, B_{G_0} *and* $\mathbf{B}_1 = B_G^2 B_{G_0}^2$, *while the numbers* $\alpha D, \alpha_0 D_0$ *and* $\alpha^2 \alpha_0^2 D^2 D_0^2$ *are the corresponding eigenvalues of* B_G, B_{G_0} *and* $\mathbf{B}_1 = B_G^2 B_{G_0}^2$, *respectively.*
(iii) If (2) is true and in addition $\widehat{A}_0, \widehat{A}$ *are densely defined then* $\mathbf{B}_1$ *is correct if and only if the number* $D_0 \neq 0$.
(iv) If (2) is valid and $\mathbf{B}_1 = B_G^2 B_{G_0}^2$ *is correct then the unique solution of the*

problem (1) is given by

$$\begin{aligned} x = \mathbf{B}_1^{-1} f &= \widehat{A}_0^{-2} \widehat{A}^{-2} f \\ &+ \frac{x_0}{\alpha_0^2 \alpha^2 D_0^2 D^2} \Big[\alpha D g \langle \widehat{A}^{-1} f, F^t \rangle_{H^m} + g \langle f, F^t \rangle_{H^m} \\ &+ y \langle \widehat{A}^{-2} f, \Phi^t \rangle_{H^m} + \alpha_0 D_0 y \langle \widehat{A}_0^{-1} \widehat{A}^{-2} f, \Phi^t \rangle_{H^m} \Big]. \quad \square \qquad (6) \end{aligned}$$

4. Applications

In what follows $H^i(\Omega)$ denotes the Sobolev space of all complex functions of $H = L_2(\Omega)$ which have generalized derivatives up to the i order that are Lebesque integrable, $i = 1, 2, 3, 4$, $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$.

Example 3. Consider the operator $\mathbf{B}_1 : L_2(\Omega) \to L_2(\Omega)$ defined by

$$\begin{aligned} \mathbf{B}_1 u &= u_{xxyy} + V \int_0^1 \int_0^1 (2x-1) u_x dy dx \\ &+ i\pi e^{-i\pi(x+y)} \int_0^1 \int_0^1 (2x-1) u_{xx} dx dy + S \int_0^1 \int_0^1 (y^2 - y) u_{xxy} dx dy \\ &- 2\pi^2 e^{-i\pi(x+y)} \int_0^1 \int_0^1 (2y-1) u_{xxy} dx dy = f, \\ D(\mathbf{B}_1) &= \{ u \in H^4(\Omega) : \ u(0, y) + u(1, y) = 0, \ u_x(0, y) + u_x(1, y) = 0, \\ & \qquad u_{xx}(x, 0) + u_{xx}(x, 1) = 0, \ u_{xxy}(x, 0) + u_{xxy}(x, 1) = 0 \}, \qquad (7) \end{aligned}$$

where V, S are unknown functions of $L_2(\Omega)$. By Theorem 2 we have:
(i) If

$$S = 2\pi i (\pi^2 - 16) e^{-i\pi(x+y)}, \quad V = \frac{i\pi^2 (\pi^2 - 32\pi + 264)}{(\pi^2 - 16)^2} e^{-i\pi(x+y)}, \qquad (8)$$

then $\mathbf{B}_1$ is correct and has the unique decomposition $\mathbf{B}_1 = B_G^2 B_{G_0}^2$, where B_{G_0}, B_G are defined by (4)-(5), respectively and $G_0 = \frac{i\pi^3}{\pi^2+16} e^{-i\pi(x+y)}$,
(ii) If S, V satisfy (8) then the number $\beta = \frac{(\pi^4+32\pi^2+264)^2}{(\pi^2+16)^2}$ is an eigenvalue of $\mathbf{B}_1$ and $x_0 = e^{-i\pi(x+y)}$ its corresponding eigenvector,
(iii) If S, V satisfy (8) then the unique solution of (7) is given by

$$\begin{aligned} x &= \mathbf{B}_1^{-1} f = \widehat{A}_0^{-2} \widehat{A}^{-2} f + \frac{x_0}{\beta} \Big[\frac{i\pi(\pi^2+16)}{6} \int_0^1 \int_0^1 (4y^3 - 6y^2 + 1) f(x, y) dx dy \\ &- 2\pi \int_0^1 \int_0^1 (y^2 - y) f(x, y) dx dy + \frac{i\pi}{2} \int_0^1 \int_0^1 (2x-1)(y - y^2) f(x, y) dx dy \\ &- \frac{\pi^2 (\pi^4 + 32\pi^2 + 264)}{2(\pi^2+16)^2} \int_0^1 \int_0^1 (x^2 - x)(y - y^2) f(x, y) dx dy \Big]. \end{aligned}$$

where

$$
\begin{aligned}
\widehat{A}_0^{-2}\widehat{A}^{-2}f \quad = \quad & \int_0^x (x-x_1)dx_1 \int_0^y (y-y_1)f(x_1,y_1)dy_1 \\
& -\frac{1}{4}\int_0^x (x-x_1)dx_1 \int_0^1 (2y-2y_1+1)f(x_1,y_1)dy_1 \\
& -\frac{1}{4}\int_0^1 (2x-2x_1+1)dx_1 \int_0^y (y-y_1)f(x_1,y_1)dy_1 \\
& +\frac{1}{16}\int_0^1 (2x-2x_1+1)dx_1 \int_0^1 (2y-2y_1+1)f(x_1,y_1)dy_1.
\end{aligned}
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References

[1] L.M. Barkovsky and A.N. Furs, *Factorization of integro-differential equations of the acoustics of dispersive viscoelastic anisotropic media and the tensor integral operators of wake packet velosities.* Acoustical Physics. 48 (2002), no. 2., 128-132.

[2] E.A. Khachatryan, *Solvability of an Integro-Differential Equation with Partial Derivatives on the Half-Axis*, Journal of Contemporary Mathematical Analysis. 43 (2008), no. 5., 60-69.

[3] I.N. Parasidis, E. Providas and P.C. Tsekrekos, *Factorization of linear operators and some eigenvalue problems of special operators.* Vestn. Bashkir. Univ. Russia. 17 (2012), no. 2., 830-839.

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