Russian Academy of Sciences
St.Petersburg Department
of Steklov Mathematical Institute
Euler International Mathematical Institute

International Conference

Polynomial Computer Algebra

International Conference on Polynomial Computer Algebra
St. Petersburg, 17–22 April, 2017

Санкт Петербург
2017

ISBN 978-5-9651-1057-5

International Conference
Polynomial Computer Algebra '2017
St. Petersburg, Russia
April 17-22, 2017
International Euler Institute

International Conference Polynomial Computer Algebra '2016; St. Petersburg, April 17-22, 2017 / Euler International Mathematical Institute, Ed. by N. N. Vassiliev, VVM Pubishing, 2016, 114 pp.

The book contains short papers, extended abstracts and abstracts of reports presented at the International Conference on Polynomial Computer Algebra 2017, St.Petersburg, April 2017.

Table of content

Two formulas of planetary motion

Semjon Adlaj

Abstract. Together the first and the second Kepler's laws (of planetary motion) imply the third. Kepler's first law might be expressed via a polar equation of an ellipse representing the orbit, with the origin (of the coordinate system) representing the Sun. The second law might be combined with the third via an elementary formula which readily applies to calculating the length of the four seasons. Although such formula would not take into account the precession and nutation of Earth's axis but relies on a single parameter (that is, the eccentricity of the orbit), it coincides (up to a proper fraction of an hour) with the observable lengths of the seasons!

The first Kepler's law might be expressed via a formula, expressing the distance r between the Sun and its orbiting planet as a function of the true anomaly θ:

$$r = r(\theta) = \frac{p}{1 + e\cos\theta},\ p := \frac{b^2}{a} = a\left(1 - e^2\right),$$

where p is the (so-called) semilatus rectum, whereas a and b are the lengths of semi-major and semi-minor axes, respectively. The ellipticity of the orbit imposes upon its eccentricity e the condition: $-1 < e < 1$.

Kepler's equation calculates the (so-called) mean anomaly M via the eccentric anomaly E as

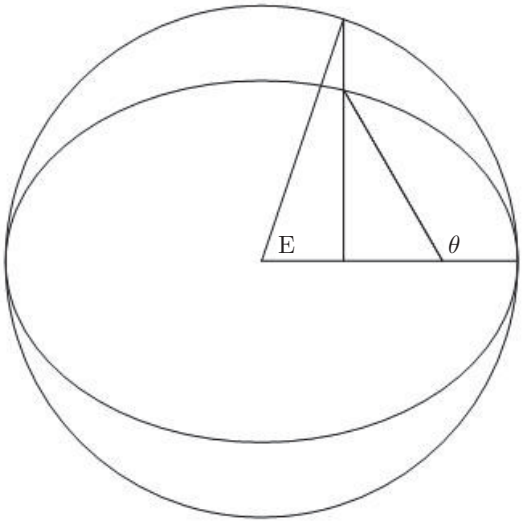
$$M = E - e\sin E.^{1}$$

We might, as well, calculate the mean anomaly M as a function of the true anomaly θ as

$$M = M(\theta) = 2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right) - \frac{e\sqrt{1-e^2}\,\sin\theta}{1 + e\cos\theta},$$

[1] At the end of the Fourth Part of his work "De Motibus Stellae Martis", Kepler states, according to a traslation from the Latin [1], concerning the solution of the problem so long known by his name (that is, concerning the determination of E for a given M):

I am sufficiently satisfied that it cannot be solved a priori, on account of the different nature of the arc and the sine. But if I am mistaken, and any one shall point out the way to me, he will be in my eyes the great Apollonius.



Eccentric and true anomalies

where the angle θ might be confined to lie in the closed interval $[-\pi, \pi]$ as the principal branch of the (multi-valued) function $\arctan(\cdot)$ is assumed. Thus, the two summands on the right hand side of the latter equation correspond to the two summands on the right hand side of the preceding Kepler's equation. One might verify the equivalence of the two formulas (for the mean anomaly M) via the accessory identities:

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E},\ \sin\theta = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E},\ \tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\,\tan\left(\frac{E}{2}\right).$$

Now, the second and the third Kepler's laws might be unified (and strengthened) via a formula for the orbiting time $t = t(\theta)$ as

$$t = \sqrt{\frac{a^3}{\mu}}\,M, \tag{1}$$

where μ is the product of the mass of the Sun with the gravitational constant.

Differentiating the latter time function t with respect to the true anomaly θ and taking the reciprocal, we readily calculate $\dot{\theta}$, thereby deriving a strong version of Kepler's second law as

$$\dot{\theta}\, r^2 = \sqrt{\mu\, p},$$

where the dot above denotes differentiation with respect to time. Moreover, integrating the latter equality, over a full period T, readily yields Kepler's third

$$\overline{\mu\, p}\, T \implies T = 2\pi \sqrt{\frac{a^3}{\mu}}.$$

Moreover, having explicitly expressed t as a function of θ, we might calculate the lengths of the four seasons on Earth. Assuming $e = 5/299$, $T = 1461/4$ and the value of the true anomaly θ at the vernal equinox is $3\pi/7$, we calculate the lengths of Winter and Spring as

$$t\left(\frac{3\pi}{7}\right) - t\left(-\frac{\pi}{14}\right) \approx 88.995,\ t\left(\frac{13\pi}{14}\right) - t\left(\frac{3\pi}{7}\right) \approx 92.765,$$

respectively. If we maintain the values of the argument θ but flip the sign of eccentricity ($e \mapsto -e$), then the two differences (upon evaluating t) would correspond to the lengths of Summer (93.651) and Autumn (89.839), respectively. Thus, the length of the "polar night" at the North (South) Pole is 178.83 (186.42).[2] These calculations closely agree with the actual lengths of seasons, which are subject to small alterations which order of magnitude matches the (20 minutes) discrepancy between the tropical and the sidereal year along with other lesser components of precession (and nutation) of the equinoxes, such as the Chandler wobble [2], which we shall further discuss at the upcoming talk.

Formula (1), where the mean anomaly M expressed via the true anomaly θ, is highly relevant for constructing "the rotating celestial sphere" [3], which pilot version was publicly first ever presented by Takayuki Ohira in Moscow, Russia, via a video communication from Yokohama, Japan, on April 2, 2017 [4].

References

[1] Hall A. Kepler's Problem // The Analyst **10** (3), May 1883: 65-66. Available at `http://www.jstor.org/stable/2635832`.

[2] Furuya M. & Chao B. F. *Estimation of period and Q of the Chandler wobble* // Geophysical Journal International **127**(3), 1996: 693-702.

[3] Adlaj S. *The rotating celestial sphere*. Available at `http://www.ccas.ru/depart/mechanics/TUMUS/Adlaj/CelSph.pdf`.

[4] Ohira T. A presentation of the rotating celestial sphere (on April 2, 2017) via a video communication at the First Open Junior Scientist Conference, Moscow, Russia. Available at `https://vk.com/public132056427`.

[2]The Winter at the North Pole does not get nearly as cold as the Summer gets at the South Pole, as consistently recorded and observed. The length of the "polar night" at the North Pole is over a week shorter than its length at the South Pole (although the small eccentricity of Earth's orbit makes it visibly indistinguishably from a circle). No temperature lower than that recorded at Mount McKinley, Alaska (−73.8 °C) was ever recorded at the Northern Hemisphere of Earth. Yet, the lowest temperature ever directly recorded at ground level on Earth is −89.2 °C (184.0 K), which occurred at the Soviet Vostok Station in Antarctica on July 21, 1983.

Semjon Adlaj

Semjon Adlaj
Division of Stability Theory and Mechanics of Controlled Systems
Computing Center of the Federal Research Center "Informatics and Control"
of the Russian Academy of Sciences
Moscow 119333, Russia
e-mail: SemjonAdlaj@gmail.com

S-consistent difference approximations to PDEs and a new such approximation to Navier-Stokes equations

P. Amodio, Yu. A. Blinkov, V. P. Gerdt and R. La Scala

The quality of a numerical solution to a partial differential equation (PDE) or to a system of PDEs obtained by the finite difference method is determined by the underlying finite difference approximation (FDA) to the PDE(s). It is a challenging problem to construct a FDA which inherits or mimics at the discrete level the fundamental properties of the original PDE(s) such as topology, symmetries, conservation laws, maximum principle, etc. Such FDAs are called *compatible* or *mimetic* (cf. [3]).

In [8], for linear PDE systems and regular (Cartesian) grids, the necessary condition for compatibility of FDA, *s(strong)-consistency*, was established. This condition admits an algorithmic verification via difference Gröbner bases. It was generalized in [4] to polynomially-nonlinear systems of PDEs, and some related aspects of difference algebra were studied in [7, 9].

Let F be a finite set of differential polynomials, $\{f = 0 \mid f \in F\}$ be the corresponding PDE system, $\tilde{F}$ be the set of difference polynomials such that set $\{\tilde{f} = 0 \mid \tilde{f} \in \tilde{F}\}$ forms a FDA to the PDE system on a chosen (regular) solution grid. Then FDA is called s-consistent if

$$(\,\forall \tilde{g} \in [\![\tilde{F}]\!]\,)\ (\,\exists g \in [\![F]\!]\,)\ [\,\tilde{g} \text{ is FDA to } g\,]$$

where $[\![\tilde{F}]\!]$ and $[\![F]\!]$ is respectively the perfect difference ideal (cf. [10]) generated by $\tilde{F}$ and the perfect differential ideal generated by $\tilde{F}$.

By applying the approach of paper [6] to the two-dimensional Navier-Stokes equations describing the unsteady motion of an incompressible viscous liquid of constant viscosity

$$\begin{cases} f_1 := & u_x + v_y = 0\,, \\ f_2 := & u_t + uu_x + vu_y + p_x - \frac{1}{\mathrm{Re}}\,\Delta\, u = 0\,, \\ f_3 := & v_t + uv_x + vv_y + p_y - \frac{1}{\mathrm{Re}}\,\Delta\, v = 0\,. \end{cases} \tag{1}$$

where (u, v) is the velocity field, p is the pressure, the constant Re is the Reynolds number, we constructed in [5] two s-consistent FDAs to (1). Below we refer to these FDAs as to FDA2 and FDA3. In [1] we showed the numerical superiority of FDA2 over two other FDAs that are not s-consistent. In doing so, we used the exact solution [11] to (1)

$$\begin{cases} u := 1 - e^{\lambda x}\cos(2\pi y)\,, \\ v := \frac{\lambda}{2\pi} e^{\lambda x}\sin(2\pi y)\,, \\ p := p_0 - \frac{1}{2}e^{2\lambda x}\,. \end{cases} \tag{2}$$

and the grid with temporal spacing τ and the spatial spacing h for both x and y.

In this talk we consider the property of s-consistency for FDAa to PDEs, describe its algorithmic verification and present a new FDA to the system (1) that we constructed in [2] in the following way. We have started with a direct difference approximation to the equations (1). In the conventional notations

$$u^n_{j,k} := u(n\tau, xj, yk), \quad v^n_{j,k} := v(n\tau, xj, yk), \quad p^n_{j,k} := p(n\tau, xj, yk)$$

for the grid functions u, v, p such approximation reads

$$\begin{aligned} &\frac{u^n_{j+1,k} - u^n_{j-1,k}}{2h} + \frac{v^n_{j,k+1} - v^n_{j,k-1}}{2h} = 0\,, \\ &\frac{u^{n+1}_{j,k} - u^n_{j,k}}{\tau} + \frac{(u^n_{j+1,k})^2 - (u^n_{j-1,k})^2}{2h} + \frac{v^n_{j,k+1}u^n_{j,k+1} - v^n_{j,k-1}u^n_{j,k-1}}{2h} + \frac{p^n_{j+1,k} - p^n_{j-1,k}}{2h} \\ &\qquad - \frac{1}{\mathrm{Re}}\left(\frac{u^n_{j+1,k} - 2u^n_{j,k} + u^n_{j-1,k}}{h^2} + \frac{u^n_{j,k+1} - 2u^n_{j,k} + u^n_{j,k-1}}{h^2}\right) = 0\,, \\ &\frac{v^{n+1}_{j,k} - v^n_{j,k}}{\tau} + \frac{(v^n_{j,k+1})^2 - (v^n_{j,k-1})^2}{2h} + \frac{u^n_{j+1,k}v^n_{j+1,k} - u^n_{j-1,k}v^n_{j-1,k}}{2h} + \frac{p^n_{j,k+1} - p^n_{j,k-1}}{2h} \\ &\qquad - \frac{1}{\mathrm{Re}}\left(\frac{v^n_{j+1,k} - 2v^n_{j,k} + v^n_{j-1,k}}{h^2} + \frac{v^n_{j,k+1} - 2v^n_{j,k} + v^n_{j,k-1}}{h^2}\right) = 0\,. \end{aligned} \tag{3}$$

Aiming to obtain a time-independent equation with linear leading monomial in the variable p in order to solve numerically the FDA, we have performed a difference Gröbner basis computation with pure lexicographic ordering with $p > u > v$ and $\partial_t > \partial_x > \partial_y$. We have obtained then a finite difference Gröbner basis consisting of 5 elements where we have found an equation of the required form

$$\begin{aligned} &\frac{p^n_{j+2,k} - 2p^n_{j,k} + p^n_{j-2,k}}{4h^2} + \frac{p^n_{j,k+2} - 2p^n_{j,k} + p^n_{j,k-2}}{4h^2} \\ &\quad + \frac{(u^n_{j+2,k})^2 - 2(u^n_{j,k})^2 + (u^n_{j-2,k})^2}{4h^2} + \frac{(v^n_{j,k+2})^2 - 2(v^n_{j,k})^2 + (v^n_{j,k-2})^2}{4h^2} \\ &\quad + 2\frac{u^n_{j+1,k+1}v^n_{j+1,k+1} - u^n_{j+1,k-1}v^n_{j+1,k-1} - u^n_{j-1,k+1}v^n_{j-1,k+1} + u^n_{j-1,k-1}v^n_{j-1,k-1}}{4h^2} \\ &+ \frac{2}{\mathrm{Re}}\frac{-u^n_{j+2,k} + 4u^n_{j+1,k} - 4u^n_{j-1,k} + u^n_{j-2,k} - u^n_{j+1,k+1} - u^n_{j+1,k-1} + u^n_{j-1,k+1} + u^n_{j-1,k-1}}{4h^3} \\ &+ \frac{2}{\mathrm{Re}}\frac{-v^n_{j,k+2} + 4v^n_{j,k+1} - 4v^n_{j,k-1} + v^n_{j,k-2} - v^n_{j+1,k+1} - v^n_{j-1,k+1} + v^n_{j+1,k-1} + v^n_{j-1,k-1}}{4h^3} = 0\,. \end{aligned} \tag{4}$$

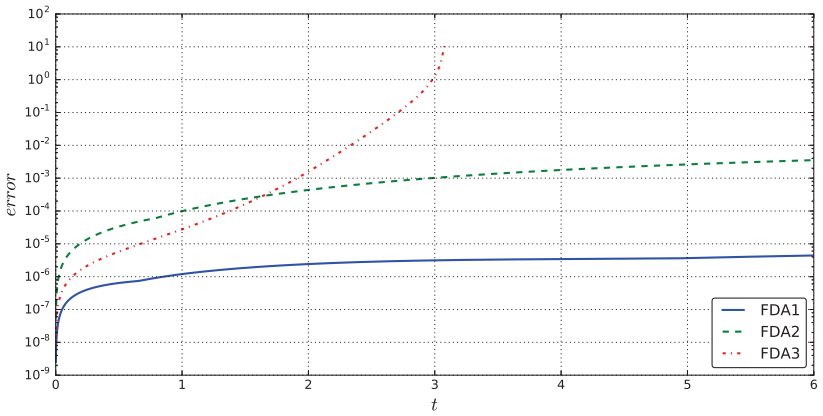


FIGURE 1. Dynamics of numerical error

It is interesting to note that this computer-generated difference equation is in fact the approximation of the following differential equation

$$(p_{xx} + p_{yy}) + 2(u_x^2 + u_x v_y + u_y v_x + v_y^2 + u(u_{xx} + v_{xy}) + v(u_{xy} + v_{yy})) + \\ - \frac{1}{\mathrm{Re}}(u_{xxx} + u_{xyy} + v_{xxy} + 6v_{yyy}) = 0.$$

One can check that this equation belongs to the differential ideal generated by the Navier-Stokes equations (1) which provides the s-consistency of a new scheme FDA1 that we have obtained by joining the equation (4) to the system (3).

Figure 1 demonstrates dynamics of the numerical errors in computation of pressure p for the three difference approximations, FDA1 and two s-consistent FDAs (FDA2 and FDA3) constructed in [5], and for the Cauchy problem with initial data taken according to (2). As one can see from the figure, the numerical behavior of FDA1 is much better than that of FDA2 and FDA3. In doing so, FDA3 reveals instability. The errors in u and v have similar behavior.

References

[1] P. Amodio, Yu.A. Blinkov, V. P. Gerdt, R. La Scala. On Consistency of Finite Difference Approximations to the Navier-Stokes Equations. *Computer Algebra in Scientific Computing / CASC 2013*, V. P. Gerdt, W. Koepff, E. W. Mayr, E. V. Vorozhtsov (Eds.), LNCS 8136, pp. 46–60. Springer, Cham, 2013. arXiv:math.NA/1307.0914

[2] P. Amodio, Yu.A. Blinkov, V. P. Gerdt, R. La Scala. Algebraic construction and numerical behavior of a new s-consistent difference scheme for the 2D Navier-Stokes equations. Submitted to Applied Mathematics and Computation.

[3] J. E. Castillo, J. F. Miranda. Mimetic Discretization Methods. CRC Press, 2013.

[4] V. P. Gerdt. Consistency Analysis of Finite Difference Approximations to PDE Systems. Mathematical *Modelling in Computational Physics / MMCP 2011*, Adam, G., Buša, J., Hnatič, M. (Eds.), LNCS 7125, pp. 28–42. Springer, Berlin (2012). arXiv:math.AP/1107.4269

[5] V. P. Gerdt, Yu.A. Blinkov. Involution and Difference Schemes for the Navier-Stokes Equations. *Computer Algebra in Scientific Computing / CASC 2009*, V. P. Gerdt, E. W. Mayr, E. V. Vorozhtsov (Eds.), LNCS 5743, pp. 94–105. Springer, Berlin, 2009.

[6] V. P. Gerdt, Yu. A. Blinkov, V. V. Mozzhilkin. Gröbner Bases and Generation of Difference Schemes for Partial Differential Equations. *Symmetry, Integrability and Geometry: Methods and Applications*, 2:26, 2006.

[7] V. P. Gerdt, R. La Scala, Noetherian quotients of the algebra of partial difference polynomials and Gröbner bases of symmetric ideals, *Journal of Algebra*, 423, 1233–1261, 2015.

[8] V. P. Gerdt, D. Robertz. Consistency of Finite Difference Approximations for Linear PDE Systems and its Algorithmic Verification). Watt, S.M. (Ed.), *Proceedings of ISSAC 2010*, pp. 53–59. Association for Computing Machinery (2010).

[9] R. La Scala, Gröbner bases and gradings for partial difference ideals, *Math. Comp.*, 84, 959–985, 2015.

[10] A. Levin. Difference Algebra. *Algebra and Applications*, Vol. 8. Springer, 2008.

[11] J. Kim, P. Moin. Application of a Fractional-Step Method To Imcompressible Navier-Stokes Equations. *J. Comput. Phys.* 59, 308–323, 1985.

P. Amodio
Dipartimento di Matematica,
Università di Bari, Bari, Italia
e-mail: `pierluigi.amodio@uniba.it`

Yu. A. Blinkov
Mechanics and Mathematics Department
Saratov State University, Saratov, Russia
e-mail: `blinkovua@info.sgu.ru`

V. P. Gerdt
Laboratory of Information Technologies
Joint Institute for Nuclear Research, Dubna, Russia
and
Peoples' Friendship University of Russia, Moscow, Russia
e-mail: `gerdt@jinr.ru`

R. La Scala
Dipartimento di Matematica,
Università di Bari, Bari, Italia
e-mail: `roberto.lascala@uniba.it`

On parametrization of nilpotent orbits of $Sp(N, \mathbb{C})$.

Mikhail V. Babich

Let $< \ldots, \ldots >$ be a non-degenerated scalar product, and $\mathfrak{G}$ be a matrix group preserving this product. Matrix A belongs to the corresponding Lie algebra $\mathfrak{g}$ iff $< A\xi, \eta > + < \xi, A\eta >= 0 \ \forall \xi, \eta$. Let ξ_1 and ξ_2 be eigenvectors corresponding to λ_1 and λ_2 correspondingly:

$$(\lambda_1 + \lambda_2) < \xi_1, \xi_2 >=< A\xi_1, \xi_2 > + < \xi_1, A\xi_2 >= 0.$$

Eigenvectors ξ_1 and ξ_2 are orthogonal if $\lambda_1 + \lambda_2 \neq 0$, particularly all eigenspaces corresponding non-zero eigenvalues are isotropic. Evidently $< (A - \lambda' \mathrm{I})\xi, \eta >= - < \xi, (A + \lambda' \mathrm{I})\eta >$ for any λ', consequently these two quadratic forms have the tame rank and if λ' is an eigenvalue , $-\lambda'$ is the eigenvalue too and $\ker^{\perp}(A - \lambda' \mathrm{I}) = \mathrm{im}(A + \lambda' \mathrm{I})$.

The method of the parametrization of the orbit $\mathcal{O}_J := \bigcup\limits_{g \in \mathfrak{G}} gJg^{-1}$ is based on the observation that (almost) any isotropic subspace can be transformed to the standard (coordinate) isotropic subspace of the proper dimension by an element of $\mathfrak{G}$

$$\Phi = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ \phi & \mathrm{I} & 0 \\ \phi_{\square} & \tilde{\phi} & \mathrm{I} \end{pmatrix},$$

in the basis with the antidiagonal Gram-matrix of $< \ldots, \ldots >$. Such Φ transform A to the form

$$A = \Phi \begin{pmatrix} \lambda' \mathrm{I} & \rho & \rho_{\square} \\ 0 & \widetilde{A} & \widetilde{\rho} \\ 0 & 0 & -\lambda' \mathrm{I} \end{pmatrix} \Phi^{-1},$$

and Lie-Poisson-Kirillov-Kostant forms calculated for A and for $\widetilde{A}$ differ on the value

$$\Delta\omega = \mathrm{tr}\, d(2\rho + \rho_{\square}\phi^{\vdash} J) \wedge d\phi + \mathrm{tr}\, d\rho_{\square} \wedge d\phi_{\square},$$

where $\phi^{\vdash}$ is the conjugation with respect to the antidiagonal[1] and $J = \mathrm{I}$ for the orthogonal groups and $J = diag(-\mathrm{I}, \mathrm{I})$ for the symplectic groups.

[1] $\phi^{\vdash} := \tau_1 \phi^T \tau_2$ where $\tau_{1,2}$ are the matrices of the inversion of the proper sizes

It is not difficult to see that if $\lambda' \neq 0$, the values ρ, ϕ and (anti)symmetrical parts of $\phi_\square$ and $\rho_\square$ are independent, consequently $P := 2\rho + \rho_\square \phi^\vdash J, Q := \phi$ and over-antidiagonal parts of $\phi_\square \pm \phi_\square^\perp$ and $\rho_\square$ form the corresponding part of the Darboux coordinates on the orbit. The zero eigenvalue is the complicated case. The eigenspace is not isotropic and this method does not work.

The theory can be adopted not for the eigenspace itself but for its isotropic subspaces. Such subspaces always exist, because $\ker A \cap \operatorname{im} A \neq 0$ and any subspace of this space is isotropic. All the speculations and the finale formula are the same. Orthogonal (or symplectic) transformation Φ moves the coordinate subspaces to the isotropic subspace of the kernel of A. The operation can be iterated, but the serious problem arise – the matrices ρ and $\rho_\square$ *are not independent* now, $\rho = \rho(\widetilde{A})$ and $\rho_\square = \rho_\square(\widetilde{A})$.

The corresponding equations can be solved in one special case, when the isotropic space that we use for the construction of Φ is the smallest one, i.e. if we use $\ker A \cap \operatorname{im} A^{M-1} \neq 0$ such that $A^M = 0$. In this case the careful consideration of the Jordan forms of $A|_{\operatorname{im} A^k} / \ker A^m$ gives the following theorem.

Let the next flight of the iteration that transform the coordinate subspace to the smallest isotropic subspace of $\widetilde{A}$ gives the following matrices $\psi, \psi_\square, \mu, \mu_\square$:

$$\widetilde{A} = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ \psi & \mathrm{I} & 0 \\ \psi_\square & \tilde{\psi} & \mathrm{I} \end{pmatrix} \begin{pmatrix} 0 & \mu & \mu_\square \\ 0 & \tilde{\tilde{A}} & \tilde{\mu} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 & 0 \\ \psi & \mathrm{I} & 0 \\ \psi_\square & \tilde{\psi} & \mathrm{I} \end{pmatrix}^{-1},$$

their matrix elements are some rational functions of the matrix elements of $\tilde{A}$ (and A of cause).

Theorem. *On the algebraically open set of the matrix elements of A, the equations connecting $\phi, \phi_\square, \rho, \rho_\square$ and $\widetilde{A}$ are*

$$\rho \begin{pmatrix} \mathrm{I} \\ \psi \\ \psi_\square \end{pmatrix} = 0,$$

here $\psi = \psi(\tilde{A})$, $\psi_\square = \psi_\square(\tilde{A})$.

It is $\zeta_M \times \zeta_{M-1}$ equations, where $\zeta_M = \dim \ker A \cap \operatorname{im} A^{M-1}$ is the number of the largest Jordan blocks ($M \times M$) and $\zeta_{M-1} = \dim \ker A \cap \operatorname{im} A^{M-2} - \dim \ker A \cap \operatorname{im} A^{M-1}$ is the number of the Jordan blocks one unit smaller ($(M-1) \times (M-1)$).

It is evident that the coordinate set is formed by the independent elements of all introduced matrix-blocks, i.e. all constructed matrices except the left $\zeta_k \times \zeta_{k-1}$ blocks of each constructed block-row (ρ, μ etc.). The *problem is to construct Darboux coordinates*, that is to combine all the independent variables in such a way that $\omega = \sum_i dp_i \wedge dq_i$. The problem is the cubic term $\operatorname{tr} d(\rho_\square \phi^\vdash J) \wedge d\phi$. If both terms $d\rho \wedge d\phi$ and $d(\rho_\square \phi^\vdash J) \wedge d\phi$ present, the sum has no necessary form. Let us consider the case when $\rho = 0$, it is the case $A^2 = 0$, there are the Jordan blocks 2×2 and eigenvectors without generalized eigenvectors. We consider the symplectic case $J = diag(-\mathrm{I}, \mathrm{I})$, the orthogonal case needs some technical work.

If $A^2 = 0$ we have only one step of the iteration:

$$\omega = \operatorname{tr} d(\rho_\square \phi^\vdash J) \wedge d\phi + \operatorname{tr} d\rho_\square \wedge d\phi_\square.$$

It is known that a matrix from the Lie algebra $sp(N)$ has an even number of the eigenvectors without the generalized eigenvectors (the Jordan blocks of the odd size 1×1). It means that matrix $\phi^\vdash$ has even number of columns, and ϕ has even number of rows. Let us denote

$$\phi = \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix}, \quad \phi^\vdash = (\phi_+^\vdash, \phi_-^\vdash).$$

$$\omega = -\operatorname{tr} d(\rho_\square \phi_+^\vdash) \wedge d\phi_- + \operatorname{tr} d(\rho_\square \phi_-^\vdash) \wedge d\phi_+ + \operatorname{tr} d\rho_\square \wedge d\phi_\square =$$
$$= \operatorname{tr} d(2\rho_\square \phi_-^\vdash) \wedge d\phi_+ + \operatorname{tr} d\rho_\square \wedge d(\phi_\square - \phi_+^\vdash \phi_-).$$

We use nice identity

$$\operatorname{tr} d(AB) \wedge dC + \operatorname{tr} d(BC) \wedge dA + \operatorname{tr} d(CA) \wedge dB = 0,$$

and the symmetry $\rho_\square^\vdash = \rho_\square$. The conjugated coordinates are symmetric matrix elements of $2\rho_\square \phi_-^\vdash$ and ϕ_+ and the symmetric matrix elements of the over-antidiagonal elements of $\rho_\square$ and $\phi_\square - \phi_+^\vdash \phi_- + (\phi_\square - \phi_+^\vdash \phi_-)^\vdash$, and the corresponding antidiagonal elements of square matrices $\rho_\square$ and $\phi_\square - \phi_+^\vdash \phi_-$.

References

[1] I. M. Gelfand and M. I. Najmark , "Unitary representation of classical groups", Trudy matemeticheskogo instituta imeni V.A.Steklova, 36, (1950). (in Russian)

[2] S. E. Derkachov and A. N. Manashov, "R-matrix and Baxter Q-operators for the noncompact SL(N;C) invariant spin chain SIGMA 2", 084 arXiv:nlin.SI/0612003, (2006).

[3] M. V. Babich and S. E. Derkachov "On rational symplectic parametrization of coajoint orbit of $GL(N)$. Diagonalizable case", Algebra and Analiz, v. 22, no. 3, p. 16-30, (2010) (in Russian).

[4] М.В.Бабич, "Бирациональные координаты Дарбу на (ко)присоединенных орбитах группы GL(N,C)", Функциональный анализ и его приложения, том 50, номер 1, с. 20-37, (2016)

[5] M.V. Babich, "On Birational Darboux Coordinates On Coadjoint Orbits Of Classical Complex Lie Groups", Записки научных семинаров ПОМИ, v. 432, p. 36-57, (2015) (http://www.pdmi.ras.ru/znsl/2015/v432.html)

Mikhail V. Babich
PDMI Steklova
St.Petersburg, Russia
e-mail: mbabich@pdmi.ras.ru

Global Parametrizations and Local Expansions of One Real Variety with Boundary

Alexander Batkhin

Abstract. An algebraic variety Ω in $\mathbb{R}^3$ is studied that plays an important role in the investigation of the normalized Ricci flow on generalized Wallach spaces related to invariant Einstein metrics. A procedure for obtaining a set of global parametric representations of the variety Ω is described, which is based on the use of the intersection of this variety with the discriminant set of an auxiliary cubic polynomial as the axis of parameterization. For this purpose, elimination theory and computer algebra are used. Obtained parametrizations are not valid at the finite set of a parameter and in this case we provide local expansions of the variety near its singular points.

Introduction

A three-parameter family of special homogeneous spaces was studied from the viewpoint of the normalized Ricci flow. In this case, the Ricci flow determines the evolution of invariant (homogeneous) Riemann metrics on the homogeneous Wallach spaces. The equation of the normalized Ricci flow is reduced to a system of two ordinary differential equations with three parameters (see [1] and the references in it for details), and the singular points of this system are associated with invariant Einstein metrics.

We investigate a set Ω, which is defined by zeroes of the polynomial $Q^*(a_1, a_2, a_3) \equiv Q(s_1, s_2, s_3) = 0$, where

$$\begin{aligned} Q \stackrel{\text{def}}{=} & (2s_1 + 4s_3 - 1)(64s_1^5 - 64s_1^4 + 8s_1^3 + 240s_1^2 s_3 - 1536s_1 s_3^2 - \\ & - 4096s_3^3 + 12s_1^2 - 240s_1 s_3 + 768s_3^2 - 6s_1 + 60s_3 + 1) - \\ & - 8s_1 s_2 (2s_1 + 4s_3 - 1)(2s_1 - 32s_3 - 1)(10s_1 + 32s_3 - 5) - \\ & - 16 s_1^2 s_2^2 (52s_1^2 + 640 s_1 s_3 + 1024 s_3^2 - 52 s_1 - 320 s_3 + 13) + \\ & + 64(2s_1 - 1)s_2^3 (2s_1 - 32s_3 - 1) + 2048 s_1 (2s_1 - 1) s_2^4. \end{aligned} \tag{1}$$

Here s_1, s_2, s_3 are the symmetric polynomials defined, respectively, as

$$s_1 = a_1 + a_2 + a_3, \quad s_2 = a_1a_2 + a_1a_3 + a_2a_3, \quad s_3 = a_1a_2a_3.$$

In [2] all singular points of Ω^* were found, and a qualitative description of all the components of it and their mutual positions was given.

A global parametrization of the variety can be useful in many cases:

- for the analysis of the intersection of a pair of varieties;
- for obtaining a local expansion of the variety at a point;
- for the visualization of varieties or their projections and so on.

In [3] a set of global rational parametrizations of the variety Ω was constructed.

Definition 1. An analytical description of the variety Ω in terms of the variables s_i is called an **s**-representation, and in terms of the variables a_i an **a**-representation.

1. Obtaining global parametrization of Ω

The procedure of obtaining global parametrization is described in details in [3]. Here we give the key points of it.

The zeroes of polynomial (1) provide the **s**-representation of Ω only when the auxiliary cubic polynomial $\chi(y)$ with the coefficients **s**

$$\chi(y) \stackrel{\text{def}}{=} y^3 - s_1y^2 + s_2y - s_3 \tag{2}$$

has three real zeroes.

Proposition 1. The polynomial $\chi(y)$ has only real searoes iff its coefficients **s** satisfy the inequality $D(\chi) \geq 0$, where

$$D(\chi) = -4s_1^3s_3 + s_1^2s_2^2 + 18s_1s_2s_3 - 4s_2^3 - 27s_3^2 \tag{3}$$

is the discriminant of (2).

The *discriminant surface* $\mathcal{D}(\chi) = \{\mathbf{s}\colon D(\chi) = 0\}$ divides the coefficient space Π of $\chi(y)$ into two domains and, as was shown in [4], it allows the polynomial parametrization

$$\mathcal{D}(\chi) : \{s_1 = 2t_1 + t_2,\ s_2 = t_1^2 + 2t_1t_2,\ s_3 = t_1^2t_2\}.$$

So, to get the parametrization of the variety Ω^* in **a** variables one has to compute the parametrization of those part of the variety Ω in **s**, where polynomial $\chi(y)$ has three real zeroes, i.e. to compute parametrization for real variety with boundary.

The variety Ω intersects the discriminant surface $\mathcal{D}(\chi)$ along three one-dimensional varieties denoted by $\mathcal{Z}_i$, $i = 1, 2, 3$. These varieties are rational curves

in the parameter space Π with parametrizations

$$\mathcal{Z}_1 : \Big\{ s_1 = 2t_1 - 1/2, \qquad s_2 = t_1^2 - t_1, \qquad s_3 = -t_1^2/2 \Big\},$$

$$\mathcal{Z}_2 : \Big\{ s_1 = -\frac{u^2+u+1}{2u}, \quad s_2 = \frac{1-4u^2(u+1)^2}{4u}, \quad s_3 = \frac{(2u^2+2u+1)^2(u+1)}{32u^2} \Big\},$$

$$\mathcal{Z}_3 : \Big\{ s_1 = \frac{16u^3-1}{2\,(8u^2-1)}, \quad s_2 = \frac{u\,\big(8u^3-3u+1\big)}{8u^2-1}, \quad s_3 = -\frac{u\,\big(16u^3-4u+1\big)}{2\,(8u^2-1)} \Big\}.$$

The varieties $\mathcal{Z}_i$, $i = 1, 2, 3$, intersect (or touch) at singular points $P_i^{(k)}$ of the set Ω (see [2, 3] for details) and they are shown in Fig. 1.

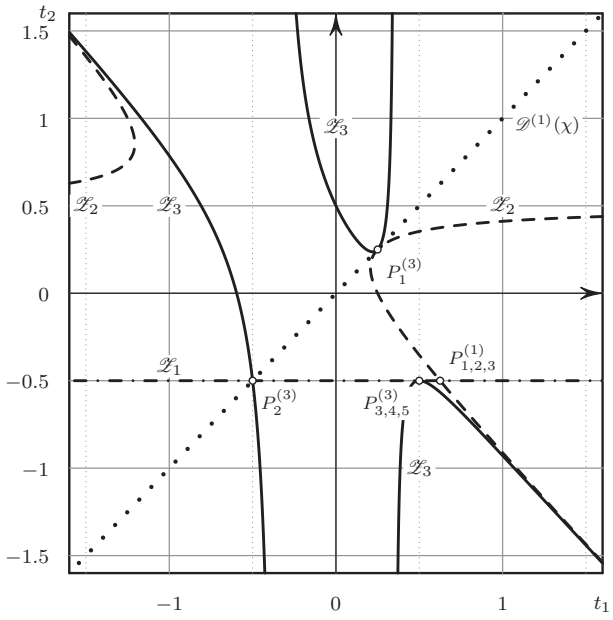


FIGURE 1. Varieties $\mathcal{Z}_i$, $i = 1, 2, 3$ and singular points $P_i^{(k)}$ on the plain (t_1, t_2).

The following approach for parametrization of the set Ω is proposed.

1. Choose one of the varieties $\mathcal{Z}_i$ and change the variables in such way that the new coordinates $\mathbf{S} = (S_1, S_2, S_3)$ give the deviation from this variety; let $\widetilde{Q}_i(\mathbf{S}) = 0$ be the equation of Ω in the new variables $\mathbf{S}$.
2. Determine the set $\mathbb{S}$ of the parameter's $S_1 = \text{const}$ values such that a curve $\mathcal{F} : \Big\{ \widetilde{Q}_i(S_2, S_3) = 0 \Big\}$ is reducible; such values are called *critical*.

3. Let for each fixed value of the parameter $S_1 \notin \mathbb{S}$ the irreducible algebraic curve $\mathcal{F}_i$ admits parametrization
$$\mathcal{F}_i : \{S_2 = \varphi(S_1, t), \quad S_3 = \psi(S_1, t)\}. \tag{4}$$
Parameter t is chosen such that for $t = 0$ vector $(S_1, \varphi(S_1, 0), \psi(S_1, 0))$ provides the parametrization of variety $\mathcal{Z}_i$.
4. Substitute parametrization (4) into (3) and find interval on which the inequality $D(S_1, t) \geq 0$ holds.

For each variety $\mathcal{Z}_i$, $i = 1, 2, 3$, the above procedure was applied and three rational parametrizations were obtained [3]. They are not valid for $s_1 \in \mathbb{S} \equiv \{-3/2, 0, 1/2, 3/4\}$ as at these values of parameter s_1 the polynomial (1) can be factorized into factors. Each of these factors gives the rational curve in the parameter space Π. Along each of these curves local expansions of the variety Ω were computed by algorithms of Power Geometry [5].

Finally, the parametrization of Ω^* in **a**-representation can be obtained with roots' formula of cubic equation for the *casus irreducibilis*.

References

[1] Z. Chen and Y. G. Nikonorov. Invariant Einstein metrics on generalized Wallach spaces. *ArXiv e-prints*, 2015. http://arxiv.org/abs/1511.02567 arXiv:1511.02567.

[2] A. B. Batkhin and A. D. Bruno. Investigation of a real algebraic surface. *Programming and Computer Software*, 41(2):74–83, 2015. URL: http://link.springer.com/article/10.1134%2FS0361768815020036, http://dx.doi.org/10.1134/S0361768815020036 doi:10.1134/S0361768815020036.

[3] A. B. Batkhin. A real variety with boundary and its global parameterization. *Programming and Computer Software*, 43(2):75–83, 2017.

[4] A. B. Batkhin. Parameterization of the discriminant set of a polynomial. *Programming and Computer Software*, 42(2):65–76, 2016. http://dx.doi.org/10.1134/S0361768816020031 doi:10.1134/S0361768816020031.

[5] A. D. Bruno and A. B. Batkhin. Resolution of an algebraic singularity by power geometry algorithms. *Programming and computer software*, 38(2):57–72, 2012. http://dx.doi.org/10.1134/S036176881202003X doi:10.1134/S036176881202003X.

Alexander Batkhin
Department of Singular Problems
Keldysh Institute of Applied Mathematics of RAS
Moscow, Russia
e-mail: batkhin@gmail.com

Estimates on the semi-meandric crossing number of classical knots

Yury Belousov and Andrei Malyutin

Abstract. A plane diagram D of a knot is said to be semi-meandric if D is the union of two simple smooth arcs. Every tame knot has a semi-meandric diagram. We use this fact to define a new knot invariant: the semi-meandric crossing number. Applying the technique of Gauss Codes and a specific algorithm transforming arbitrary knot diagrams to semi-meandric ones we obtain estimates on this invariant.

Introduction

Definition 1. A smooth closed plane curve is called *semi-meandric* if it is the union of two simple smooth arcs (an arc is called *simple* if it is non-self-intersecting).

Theorem 1. *Every tame knot has a semi-meandric diagram.*

Proof. It is shown in [1] that a knot which is dyed with two different colours can be projected on a plane without crossing strands of the same colour. Clearly, such projection gives us a semi-meandric diagram. □

Remark 1. The approach developed in [1] is based on braid theory. We use a different approach, which gives another proof of Theorem 1.

Theorem 1 allows us to define a new knot invariant: the semi-meandric crossing number.

Definition 2. Recall that the *crossing number* of a knot K (denoted $\mathrm{Cr}(K)$) is the smallest number of crossings in any diagram of K. We define the *semi-meandric crossing number* of K (denoted $\mathrm{Cr_s}(K)$) as the smallest number of crossings in any semi-meandric diagram of K.

The reported study was funded by RFBR according to the research project n. 17-01-00128 A.

Theorem 2. *For each tame knot K, the following inequalities hold*

$$\mathrm{Cr}(K) \leq \mathrm{Cr_s}(K) < (\sqrt{3})^{\mathrm{Cr}(K)}.$$

Our proof of Theorem 2 is based on properties of Gauss Codes and on the algorithm presented below, which transforms arbitrary knot diagrams to semi-meandric ones.

Obtaining semi-meandric diagrams

Given a diagram D of a knot K, we obtain a semi-meandric diagram of K using the following algorithm.

1. We choose a simple arc J in D such that no endpoint of J is an intersection point of D (the interior of J is allowed to contain intersection points of D).
2. We choose an endpoint s of J and start walking along D from s until we find the first intersection point x of D that does not belong to J. (We may intersect J while traveling from s to x.) Denote the path connecting s and x by $[s, x]$.
3. We transform D by „pulling“ x along $[s, x]$. This transformation decreases the number of intersection points that are not in J, while new intersection points of D appear on J.
4. Until our diagram is not semi-meandric, repeat steps 2 and 3.

Observe that each time, when choosing an endpoint s of J at step 2, we have two possibilities. So, if D has n intersection points (before the first transformation) and J contains m of them, then we have 2^{n-m} possible final states of our diagram. If we use greedy algorithm and minimize the number of intersection points each time at step 2, then properties of Gauss Codes imply that double implementation of above procedure at most triples the number of intersection points on J. In other words, we prove that amongst four possible ways of decreasing by two the number of intersection points that are not in J, there exists at least one way where the number of intersection points of D in J at most triples.

Gauss Codes

Definition 3. The *Gauss Code* of a diagram D of a knot is obtained as follows:

- Label the intersection points of D with integers from 1 to n, where n is the number of this points.
- Start „walking“ along D, taking note of the labels of the intersection points we have gone through. If in a given intersection point we cross on the „over“ strand, write down the label of this point. Otherwise, we write down the negative of the label of the intersection point.

Remark 2. In terms of Gauss Codes, we can give an alternative definition of semi-meandric diagrams. A diagram with n intersection points is semi-meandric if and

only if its cycled Gauss Code splits in two „connected“ parts such that the absolute values of elements in each part is precisely all numbers from 1 to n.

Remark 3. The algorithm from the previous section can be rewritten in terms of Gauss Codes. This allows us to make its exact implementation.

References

[1] Jan A. Kneissler, *Woven braids and their closures.* Journal of Knot Theory and Its Ramifications **8**:2 (1999), 201–214.

[2] Ljiljana Radovic, Slavik Jablan, *Meander knots and links.* Filomat **29**:10 (2015), 2381–2392.

Yury Belousov
St. Petersburg State University
St. Petersburg, Russia
e-mail: bus99@yandex.ru

Andrei Malyutin
St. Petersburg Department of Steklov Institute of Mathematics
St. Petersburg, Russia
e-mail: malyutin@pdmi.ras.ru

Algorithmically generated implicit difference schemes for KdV equation and their numerical behaviour

Yu. A. Blinkov, V. P. Gerdt and K. V. Marinov

• **Introduction** • In this paper we consider the Korteveg-de Vries (KdV) equation in the following form (cf. [1], Eq. 1.18)

$$\partial_t u + \alpha u\, \partial_x u + \beta\, \partial_x^3 u = 0 \tag{1}$$

where $u := u(x,t)$ and $\alpha, \beta \in \mathbb{R}$. Numerical solving of Eq. (1) by the finite difference method was intensively studied in the literature (see book [1] and its bibliography). In so doing, a number of explicit and implicit difference schemes were derived and used for numerical construction of various solutions to (1).

In our talk we consider the Cartesian grid with spacings $\tau := t_{n+1} - t_n$ and $h := x_{j+1} - x_j$ and present two new implicit schemes for Eq. (1) with $\mathcal{O}(\tau^2, h^2)$ and $\mathcal{O}(\tau^2, h^2)$ approximations. The schemes were generated by our algorithmic approach [2] which is based on combination of the methods of finite volumes, numerical integration and difference elimination by means of Gröbner bases. Then we compare, on the exact soliton solution, numerical behavior of these schemes with that of some *classical* schemes of same order of approximation used in the literature and show that our schemes provide substantially better numerical accuracy.

• **Classical schemes with $\mathcal{O}(\tau^2, h^2)$ approximation** • The explicit scheme [1], Eq.1.80

$$u_i^{n+1} = u_i^{n-1} - \frac{\alpha\tau}{h} u_i^n \left(u_{i+1}^n - u_{i-1}^n\right) - \frac{\beta\tau}{h^3}\left(u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n\right). \tag{2}$$

where the standard notation $u_j^n := u(t_n, x_j)$ for the grid function is used. This scheme is stable for

$$\tau \le \frac{2h^3}{3\sqrt{3}\beta} \cong 0.384\frac{h^3}{\beta}.$$

The work was partially supported by the grant No.16-01-00080 from the Russian Foundation for Basic Research.

The implicit scheme [1], Eq.1.96

$$\begin{aligned}\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{\alpha}{4h}\left[u_j^n\left(u_{j+1}^{n+1}-u_{j-1}^{n+1}\right)+u_j^{n+1}\left(u_{j+1}^n-u_{j-1}^n\right)\right]+\\ +\frac{\beta}{4h^3}\left(\left(u_{j+2}^{n+1}-2u_{j+1}^{n+1}+2u_{j-1}^{n+1}-u_{j-2}^{n+1}\right)+\right.\\ \left.+\left(u_{j+2}^n-2u_j^{n+1}+2u_{j-1}^n-u_{j-2}^n\right)\right)=0\,.\end{aligned} \tag{3}$$

• **Classical schemes with $\mathcal{O}(\tau^2,h^4)$ approximation** • The explicit scheme ([1], Eq.1.82)

$$\begin{aligned}u_i^{n+1}=&u_i^{n-1}-\frac{\alpha\tau}{6h}u_i^n\left(u_{i+2}^n-8u_{i+1}^n+8u_{i-1}^n-u_{i-2}^n\right)\\ &-\frac{\beta\tau}{4h^3}\left(u_{i+3}^n-8u_{i+2}^n+13u_{i+1}^n-13u_{i-1}^n+8u_{i-2}^n-u_{i-3}^n\right)\end{aligned} \tag{4}$$

whose stability condition is given by

$$\tau\le\frac{108h^3}{(43+7\sqrt{73})\sqrt{10\sqrt{73}-62}\beta}\cong 0.216\frac{h^3}{\beta}\,.$$

The implicit scheme [1], Eq.1.84

$$\begin{aligned}\frac{u_j^{n+1}-u_j^n}{\tau}=\frac{\alpha}{4h}\left[u_j^n\left(u_{j+2}^{n+1}-8u_{j+1}^{n+1}+8u_{j-1}^{n+1}-u_{j-2}^{n+1}\right)+\right.\\ \left.+u_j^{n+1}\left(u_{j+2}^n-8u_{j+1}^n+8u_{j-1}^n-u_{j-2}^n\right)\right]+\\ +\frac{\beta}{4h^3}\left(\left(u_{j+3}^{n+1}-8u_{j+2}^{n+1}+13u_{j+1}^{n+1}-13u_{j-1}^{n+1}+8u_{j-2}^{n+1}-u_{j-3}^{n+1}\right)+\right.\\ \left.\left(u_{j+3}^n-8u_{j+2}^n+13u_{j+1}^n-13u_{j-1}^n+8u_{j-2}^n-u_{j-3}^n\right)\right)\,.\end{aligned} \tag{5}$$

• **A new scheme with $\mathcal{O}(\tau^2,h^2)$ approximation** • It is obtained by the straightforward extension of the approach of paper [3] to equation (1). First, we convert (1) into the integral form

$$\oint_\Gamma\left(-\frac{\alpha}{2}u^2-\beta u_{xx}\right)dt+u\,dx=0 \tag{6}$$

valid for any simply connected integration contour Γ. Second, we chose the rectangular integration contour shown in Fig. stencil as a "control volume". Then we

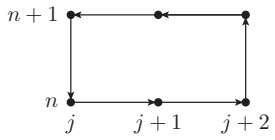


FIGURE 1. First integration contour for Eq. (6).

add to (6) the integral relations

$$\int_{x_j}^{x_{j+1}} u_x\,dx = u(t,x_{j+1}) - u(t,x_j), \int_{x_j}^{x_{j+2}} u_{xx}\,dx = u_x(t,x_{j+2}) - u_x(t,x_j). \quad (7)$$

To approximate the contour integral, we apply the trapezoidal rule. For numerical approximations of the integral relations we apply the trapezoidal rule for the integration of u_x and the midpoint rule for the integration of u_{xx}. Thereby, Eqs. (6) and (7) take the form

$$\begin{aligned}&\Big[-\frac{\alpha}{2}\left({u^2}_j^n + {u^2}_j^{n+1} - {u^2}_{j+2}^n - {u^2}_{j+2}^{n+1}\right) - \\ &- \beta\left({u_{xx}}_j^n + {u_{xx}}_j^{n+1} - {u_{xx}}_{j+2}^n - {u_{xx}}_{j+2}^{n+1}\right)\Big]\cdot\frac{\tau}{2} + \left[u_{j+1}^{n+1} - u_{j+1}^n\right]\cdot 2h = 0, \quad (8)\\ &\left[{u_x}_{j+1}^n + {u_x}_j^n\right]\cdot\frac{h}{2} = u_{j+1}^n - u_j^n, {u_{xx}}_{j+1}^n\cdot 2h = {u_x}_{j+2}^n - {u_x}_j^n.\end{aligned}$$

To use linear difference elimination of u_x and u_{xx} from system (8), and hence the Maple package LDA (Linear Difference Algebra) [4], we introduce the new function $F := \frac{\alpha}{2}u^2$ and chose an elimination ranking $\succ$ such that $u \succ F \succ u_x \succ u_{xx}$. Then computation of a differenceGröbner basis of the ideal generated by the left-hand sides of difference polynomials in (8) and extraction from the basis an equation that does not contains u_x and u_{xx} yields the following difference scheme

$$\begin{aligned}&\frac{u_j^{n+1}-u_j^n}{\tau} + \frac{\alpha}{8h}\left[\left({u^2}_{j+1}^{n+1} - {u^2}_{j-1}^{n+1}\right) + \left({u^2}_{j+1}^n - {u^2}_{j-1}^n\right)\right] + \\ &+ \frac{\beta}{4h^3}\left[\left(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}\right) + \right. \\ &\left. + \left(u_{j+2}^n - 2u_j^{n+1} + 2u_{j-1}^n - u_{j-2}^n\right)\right] = 0.\end{aligned} \quad (9)$$

This scheme is an analog of the famous Crank-Nicolson scheme for the heat equation.

• **A new scheme with $\mathcal{O}(\tau^2, h^2)$ approximation** • Choose now the integration contour Γ in (6) shown in Fig. 2 with the indentation $h/4$ along to the $x-$direction.

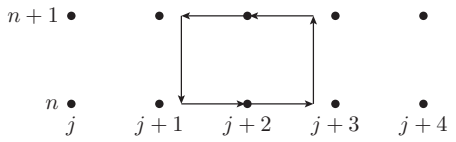


FIGURE 2. Second integration contour for Eq. (6).

Using at a fractional point relations

$$\begin{aligned}f_{j+5/4} &\approx f_{j+1} + \frac{f_{j+2}-f_j}{2h}\cdot\frac{1}{4h} = (f_{j+2} + 8f_{j+1} - f_j)/8,\\ f_{j+11/4} &\approx f_{j+3} - \frac{f_{j+4}-f_{j+2}}{2h}\cdot\frac{1}{4h} = (-f_{j+4} + 8f_{j+3} + f_{j+2})/8.\end{aligned} \quad (10)$$

we rewrite (6), (7) in the following form

$$\Big[-\frac{\alpha}{2}\Big((-u^2{}_j^n + 8u^2{}_{j+1}^n + u^2{}_{j+2}^n)/8 + (-u^2{}_j^{n+1} + 8u^2{}_{j+1}^{n+1} + u^2{}_{j+2}^{n+1})/8 - \\ - (u^2{}_{j+2}^n + 8u^2{}_{j+3}^n - u^2{}_{j+4}^n)/8 - (u^2{}_{j+2}^{n+1} + 8u^2{}_{j+3}^{n+1} - u^2{}_{j+4}^{n+1})/8\Big) - \\ - \beta\big((-u_{xx}{}_j^n + 8u_{xx}{}_{j+1}^n + u_{xx}{}_{j+2}^n)/8 + (-u_{xx}{}_j^{n+1} + 8u_{xx}{}_{j+1}^{n+1} + u_{xx}{}_{j+2}^{n+1})/8 - \\ - (u_{xx}{}_{j+2}^n + 8u_{xx}{}_{j+3}^n - u_{xx}{}_{j+4}^n)/8 - (u_{xx}{}_{j+2}^{n+1} + 8u_{xx}{}_{j+3}^{n+1} - u_{xx}{}_{j+4}^{n+1})/8)\big)\Big] \cdot \frac{\tau}{2} + \\ + \big[u_{j+1}^{n+1} - u_{j+2}^n\big] \cdot \frac{3h}{2} = 0\,. \\ \big[u_{x}{}_{j+1}^n + u_{x}{}_j^n\big] \cdot \frac{h}{2} = u_{j+1}^n - u_j^n,\ u_{xx}{}_{j+1}^n \cdot 2h = u_{x}{}_{j+2}^n - u_{x}{}_j^n. \qquad (11)$$

Elimination the grid functions u_x and u_{xx} from (11) gives the difference scheme

$$\frac{u_j^{n+1} - u_j^n}{\tau} - \frac{\alpha}{48h}\Big[\Big(u^2{}_{j+2}^{n+1} - 8u^2{}_{j+1}^{n+1} + 8u^2{}_{j-1}^{n+1} - u^2{}_{j-2}^{n+1}\Big) + \\ + \Big(u^2{}_{j+2}^n - 8u^2{}_{j+1}^n + 8u^2{}_{j-1}^n - u^2{}_{j-2}^n\Big)\Big] + \\ - \frac{\beta}{16h^3}\big[\big(u_{j+3}^{n+1} - 8u_{j+2}^{n+1} + 13u_{j+1}^{n+1} - 13u_{j-1}^{n+1} + 8u_{j-2}^{n+1} - u_{j-3}^{n+1}\big) + \\ \big(u_{j+3}^n - 8u_{j+2}^n + 13u_{j+1}^n - 13u_{j-1}^n + 8u_{j-2}^n - u_{j-3}^n\big)\big] = 0 \qquad (12)$$

Numerical results

Our numerical analysis of schemes (2)–(5), (9) and (12) was done with the Python package SciPy (`http:\\scipy.org`). as a benchmark, we used the exact one-soliton solution to (1)

$$u(x,t) = \frac{2k_1^2}{\cosh(k_1(x - 4k_1^2 t))^2}$$

with $\alpha = 6$, $\beta = 1$ and $k_1 = 0.4$. In so doing, we fixed $h = 0.25$ and considered the solution in interval $-50 \le x \le 50$ with periodic boundary conditions (cf. [1], p.49). The numerical inaccuracy was estimated by the Frobenius norm ($\|A\|_F$).

For the implicit schemes (9) and (12) we applied linearization

$$v_{k+1}^2 = v_{k+1}^2 - v_k^2 + v_k^2 = (v_{k+1} - v_k)(v_{k+1} + v_k) + v_k^2 \approx v_{k+1} \cdot 2v_k - v_k^2.$$

References

[1] V. Yu. Belashov and S. V. Vladimirov: Solitary Waves in Dispersive Complex Media: Theory, Simulation, Applications. Springer Series in Solid-State Sciences, Vol. 149, Springer-Verlag, Berlin (2005).

[2] V. P. Gerdt, Yu. A. Blinkov and V. V. Mozzhilkin: Gröbner Bases and Generation of Difference Schemes for Partial Differential Equations. Symmetry, Integrability and Geometry: Methods and Applications, 2:26, 2006. arXiv:math.RA/0605334

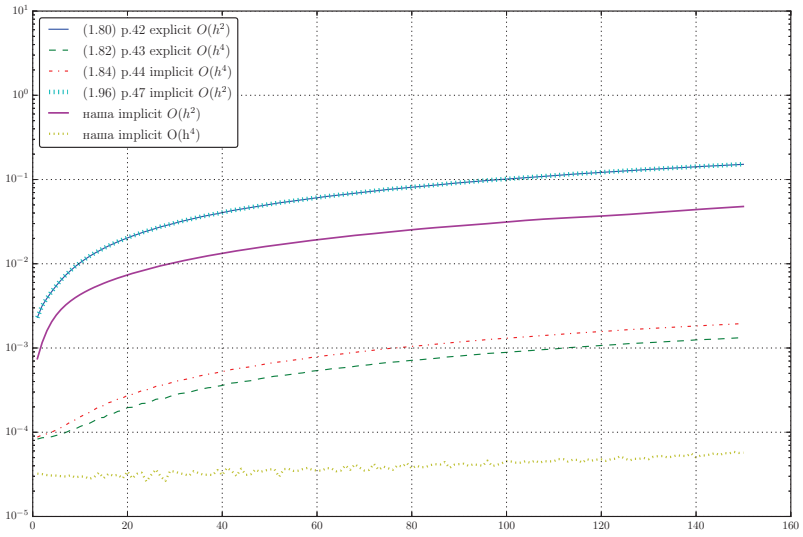


FIGURE 3. Dynamics of numerical error

[3] V. P. Gerdt, Yu. A. Blinkov and K. B. Marinov: Computer Algebra Based Discretization of Quaslinear Evolution Equations. Programming and Computer Software, to appear.

[4] V. P. Gerdt and D. Robertz: Computation of Difference Gröbner Bases. Computer Science Journal of Moldova, 20(2), (2012), 203–226. arXiv:cs.SC/1206.3463

Yu. A. Blinkov
Mechanics and Mathematics Department
Saratov State University, Saratov, Russia
e-mail: blinkovua@info.sgu.ru

V. P. Gerdt
Laboratory of Information Technologies
Joint Institute for Nuclear Research, Dubna, Russia
and
Peoples' Friendship University of Russia, Moscow, Russia
e-mail: gerdt@jinr.ru

K. V. Marinov
University "Dubna", Dubna, Russia
e-mail: marinov.kohctahtih@gmail.com

Solving the Polynomial Equations by Algorithms of Power Geometry

Alexander Bruno

Abstract. New methods for computation of solutions of an algebraic equation of three variables near a critical point are proposed. These methods are: Newton polyhedron, power transformations, new versions of the implicit function theorem and uniformization of a planar algebraic curve. We begin from a survey of the new methods of computation of solutions of an algebraic equation of one and of two variables by means of the Hadamard polygon and Hadamard polyhedron. That approach works for differential equations (ordinary and partial) as well.

Introduction

We consider the polynomial equation $p(X) = 0$, where $X = (x_1, \dots, x_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$ and coefficients of p are from $\mathbb{R}$ or $\mathbb{C}$. We search its global solution in the whole space as well as its local solutions near its singular points X^0.

1. Global solutions

1a) If $n = 1$, solutions are several points. Using the Hadamard open polygon [1], [2, Part I, Ch. IV, Sect. 2.1], we obtain truncated equations $\check{p}_j(X) = 0$, $j = 1, \dots, m$, which are easily solved, and they give approximated roots of the initial polynomial $p(X)$. We can compute roots more precisely by the Newton method.

1b) If $n = 2$, solutions form an algebraic curve f. Sometimes solutions to the equation $p(X) = 0$ are known as functions $x_1 = \varphi(t)$, $x_2 = \psi(t)$. It is so, if the genus of the curve f is not big or if the polynomial $p(X)$ has rather big group of symmetries (i.e. birational automorphisms). Using the Hadamard polyhedron [3], we divide the space into m several pieces W_j, $j = 1, \dots, m$, and find the truncated simple polynomials $\check{p}_j(X)$, $j = 1, \dots, m$, which are the main parts of the initial polynomial $p(X)$ in the corresponding pieces W_j. Usually the truncated polynomial $\check{p}_j(X)$ has a lot of symmetries and it has an uniformization $x_1 = \varphi_j(t)$, $x_2 = \psi_j(t)$

of its roots. It gives the approximate parametrization of the curve f in the piece W_j. That parametrization can be made more accurate by the Newton method. So, we obtain m different approximate uniformizations in m pieces W_j.

1c) If $n > 2$ we can apply the similar method and can obtain several parametrizations in several pieces W_j of the whole space.

2. Local solutions

Near the critical point X^0 we can construct the Newton polyhedron [4] and can obtain several truncated polynomials $\hat{p}_j(X)$. Considering their solutions as the first approximations, we can continue them as the asymptotic expansions for each branch. Sometimes for that we need the global solutions of a polynomial equation with dimension $n' < n$ and transformations of the solutions into coordinate subspaces.

That approach works for *differential equations* (ordinary and partial) as well. In the survey [5] there are several nontrivial applications.

References

[1] J. Hadamard, *Etudes sur les proprietes des fonctions entiers et en particulier d'une fonction consideree par Riemann,* J. math. pure & appliques, (1893) **9**:2, 171–215.

[2] A. D. Bruno, *Local Method of Nonlinear Analysis of Differential Equations.*Nauka: Moscow, 1979. 256 p. (in Russian) = *Local Methods in Nonlinear Differential Equations.* Springer–Verlag: Berlin-Heidelberg-New York-London-Paris-Tokyo, 1989. 350 p.

[3] A. D. Bruno, *On solution of an algebraic equation,* Preprint of KIAM, No. 70, Moscow, 2016. 20 p. (R) doi:10.20948/prepr-2016-70 `http://keldysh.ru/papers/2016/prep2016_70.pdf`

[4] A. D. Bruno, *Power Geometry in Algebraic and Differential Equations.* Moscow, Fizmatlit, 1998. 288 p. (in Russian) = *Power Geometry in Algebraic and Differential Equations.* Elsevier Science (North-Holland), Amsterdam, 2000. 385 p.

[5] A. D. Bruno, *Asymptotic solution of nonlinear algebraic and differential equations,* International Mathematical Forum, **10**:11 (2015), 535–564. `http://dx.doi.org/10.12988/imf.2015.5974`

Alexander Bruno
Department of Singular Problems
Keldysh Institute of Applied Mathematics of RAS
Moscow, Russia
e-mail: `abruno@keldysh.ru`

Further Algebraization of Differential Equations and Applications

Alexander Bruno

Abstract. We propose a method of calculation of differential equations for coefficients of the complicated expansions of solutions to a differential equation. The method uses the higher variations. It was applied to the Painlevé equations.

1. General situation

The Taylor series gives expansions of values of a polynomial using higher derivatives. I propose to use similar expansions of values of a differential polynomial using higher variations, i.e. the Freche/Gato derivatives.

2. Ordinary differential equations

Some ordinary differential equations have asymptotic expansions of solutions as a power series which coefficients are the Laurent series of logarithms (so called complicated expansions or psi-series). The mentioned approach allows to calculate differential equations for each such coefficient separately using higher variations of parts of the initial equation. These equations for coefficients can be solved by the known methods. But sometimes the coefficients are polynomials instead of to be Laurent series.

3. Painlevé equations

Among six Painlevé equations three of them have such complicated expansions of solutions: P_3, P_5 and P_6. Their first coefficients are polynomials of logarithms. In examples 3 and 4 of [1] there were computed some second coefficients for P_3 and P_6. They are polynomials. Recently I computed all second coefficients for P_3 and P_6. All of them are polynomials. Now I computed the third coefficients for P_3 and

P_6. They are polynomials only under some restrictions on four parameters of the equations in form of polynomial equations. Sometimes the third coefficients are not polynomial at all.

References

[1] A. D. Bruno, *On complicated expansions of solutions to ODE,* Preprint of KIAM No. 15. Moscow, 2011. 26 p. (Russian)

Alexander Bruno
Department of Singular Problems
Keldysh Institute of Applied Mathematics of RAS
Moscow, Russia
e-mail: abruno@keldysh.ru

Efficient solving systems of polynomial equations with parametric coefficients

Alexander L. Chistov

St. Petersburg Department of Steklov Mathematical Institute
of the Academy of Sciences of Russia
Fontanka 27, St. Petersburg 191023, Russia,
e-mail: alch@pdmi.ras.ru

2017

Abstract

Consider a system of polynomial equations with parametric coefficients over an arbitrary ground field. We show that the variety of parameters can be represented as union of strata. Each stratum is a quasiprojective algebraic variety with the degree bounded from above by a subexponential function in the size of the input data. Also the number of strata is subexponential in the size of the input data. This solves a long standing problem to avoid double exponential growth of coefficients for this problem.

Let k be an arbitrary field containing sufficiently many elements with the characteristic exponent p. Let $\nu \geqslant 0$ be an integer. Let $a_1, \ldots, a_\nu$ be a family of independent variables (or parameters) over k. Denote by $\mathbb{A}^\nu(\overline{k})$ the affine space of parameters with the coordinate functions $a_1, \ldots, a_\nu$ (in a more general situation one can consider an algebraic variety of parameters $\mathcal{V} \subset \mathbb{A}^\nu(\overline{k})$ but this case is easily reduced to the particular one: $\mathcal{V} = \mathbb{A}^\nu(\overline{k})$).

Let $m, n \geqslant 1$ be integers. Let $f_0, \ldots f_{m-1} \in k[a_1, \ldots, a_\nu, X_0, \ldots, X_n]$ be homogeneous with respect to $X_0, \ldots, X_n$ polynomials. Assume that the degrees

$$\deg_{X_1,\ldots,X_n} f_i = d_i \leqslant d, \quad \deg_{a_1,\ldots,a_\nu} f \leqslant d' \tag{1}$$

for some integers $d_0, \ldots, d_{m-1} \geqslant 0$ and $d, d' \geqslant 2$. Let $a^* = (a_1^*, \ldots, a_\nu^*) \in \mathbb{A}^\nu(\overline{k})$ Denote by $V_{a^*} \subset \mathbb{P}^n(\overline{k})$ the variety of all the solutions of the system of polynomial equations

$$f_0(a_1^*, \ldots, a_\nu^*, X_0, \ldots, X_n) = \ldots = f_{m-1}(a_1^*, \ldots, a_\nu^*, X_0, \ldots, X_n) = 0.$$

Let $-1 \leqslant c \leqslant n$ be an integer. Denote by $\mathcal{U}_c$ the subset of all $a^* \in \mathbb{A}^\nu(\overline{k})$ such that the dimension $\dim V_{a^*} \leqslant c$. One can prove that it is an open in the Zariski topology subset of $\mathbb{A}^\nu(\overline{k})$. For every point $a^* \in \mathcal{U}_c$ for every integer $0 \leqslant s \leqslant c$ denote by $V_{a^*,s}$ the union of all irreducible components W of the variety V_{a^*} such that the dimension $\dim W = s$.

Key words and phrases: parametric coefficients, stratifications, absolutely irreducible components, solving polynomial systems.

UDK 513.6+518.5; 2010 Mathematics Subject Classification: 14Q15.

Consider the problem to represent the set of parameters

$$\mathcal{U}_c = \bigcup_{\alpha\in A} \mathcal{W}_\alpha \tag{2}$$

as a union of a finite number, i.e., $\#A < +\infty$, of quasiprojective algebraic varieties $\mathcal{W}_\alpha$ satisfying the following properties. For every $\alpha \in A$ for all $a^* = (a_1^*, \dots, a_\nu^*) \in \mathcal{W}_\alpha$ the variety of solutions V_{a^*} is given uniformly, i.e., by some algebraic formulas (similarly to [2], see below for details) everywhere defined on $\mathcal{W}_\alpha$ and depending on $a_1^*, \dots, a_\nu^*$ as parameters.

For an arbitrary polynomial $f \in k[a_1, \dots, a_\nu, X_0, \dots, X_n]$ and a point $a^* = (a_1^*, \dots, a_\nu^*) \in \mathbb{A}^\nu(\overline{k})$ we shall write $f(a^*, X_0, \dots, X_n) = f(a_1^*, \dots, a_\nu^*, X_0, \dots, X_n)$ and use other similar notations. We shall write $V_{a^*} = \mathcal{Z}(f_i(a^*, X_0, \dots, X_n),\ 0 \leqslant i \leqslant m-1)$. Here $\mathcal{Z}(f_i(a^*, X_0, \dots, X_n),\ 0 \leqslant i \leqslant m-1)$ denotes the set of all common zeroes of the considered polynomials in $\mathbb{P}^n(\overline{k})$. We will use also other analogous notations. In what follows all the constants in $O(\dots)$ are absolute. All the linear forms $Y_i, Y_{i,v}$ from the described below construction can be chosen with coefficients of length, say, $O(n^2 \log_2 d)$ in any subring of k with sufficiently many elements.

Now we are going to give the precise meaning to the uniformity of algebraic formulas related to (2). Namely the following properties hold true.

(i) For every $\alpha \in A$ the variety $\mathcal{W}_\alpha \neq \varnothing$. For all $\alpha_1, \alpha_2 \in A$ if $\alpha_1 \neq \alpha_2$ then $\mathcal{W}_{\alpha_1} \cap \mathcal{W}_{\alpha_2} = \varnothing$, i.e., these varieties $\mathcal{W}_\alpha$ are pairwise disjoint; so we shall call them strata and union (2) is a stratification.

(ii) One can represent

$$\mathcal{W}_\alpha = \mathcal{W}_\alpha^{(1)} \setminus \bigcup_{2\leqslant\beta\leqslant\mu_\alpha} \mathcal{W}_\alpha^{(\beta)}$$

where each $\mathcal{W}_\alpha^{(\beta)} = \mathcal{Z}(\psi_{\alpha,1}^{(\beta)}, \dots, \psi_{\alpha,m_{\alpha,\beta}}^{(\beta)})$, $1 \leqslant \beta \leqslant \mu_\alpha$, is the set of all common zeroes of the polynomials $\psi_{\alpha,1}^{(\beta)}, \dots, \psi_{\alpha,m_{\alpha,\beta}}^{(\beta)} \in k[a_1, \dots, a_\nu]$ in the affine space $\mathbb{A}^\nu(\overline{k})$, $m_{\alpha,\beta} \geqslant 1$ is an integer.

For every $\alpha \in A$ for every integer $0 \leqslant s \leqslant c$ there are linear forms $Y_0, \dots, Y_{s+1} \in k[X_0, \dots, X_n]$ (depending on α and s) satisfying the following properties.

(iii) For every $a^* \in \mathcal{W}_\alpha$ the intersection $V_{a^*,s} \cap \mathcal{Z}(Y_0, \dots, Y_s) = \varnothing$ in $\mathbb{P}^n(\overline{k})$.

(iv) Let $\rho = 0$ if $p = 1$ and otherwise $\rho = \log_p d^{n-s}$. For every integer $0 \leqslant r \leqslant \rho$ there is a nonzero polynomial $\Phi_{\alpha,s,r} \in k[a_1, \dots, a_\nu, Y_0, \dots, Y_{s+1}]$ homogeneous with respect to $Y_0, \dots, Y_{s+1}$ such that for every $a^* \in \mathcal{W}_\alpha$ the degree

$$0 \leqslant \deg_{Y_0,\dots,Y_{s+1}} \Phi_{\alpha,s,r} = \deg_{Y_{s+1}} \Phi_{\alpha,s,r}(a^*, Y_0, \dots, Y_{s+1}) \leqslant d^{n-s}/p^r,$$

the leading coefficient $\mathrm{lc}_{Y_{s+1}} \Phi_{\alpha,s,r} \in k[a_1, \dots, a_\nu]$, and the polynomial $\prod_{0\leqslant r\leqslant\rho} \Phi_{\alpha,s,r}(a^*, Y_0^{p^r}, \dots, Y_{s+1}^{p^r})$ vanishes on the projective algebraic variety $V_{a^*,s}$. Finally denote by $\Delta_{\alpha,s,r}$ the discriminant of the polynomial $\Phi_{\alpha,s,r}$ with respect to Y_{s+1} (by definition $\Delta_{\alpha,s,r} = 1$ if $\deg_{Y_{s+1}} \Phi_{\alpha,s,r} = 0$). Then for every $a^* \in \mathcal{W}_\alpha$ the polynomial $\Delta_{\alpha,s,r}(a^*, Y_0, \dots, Y_s) \neq 0$.

mials $H_j \in k[a_1, \ldots, a_\nu, Z]$, $j \in J_{\alpha,s,r}$, satisfying the following properties. The degree $1 \leqslant \deg_Z H_j \leqslant d^{n-s}$. Denote by Δ_j the discriminant of the polynomial Δ_j with respect to Z. Then $\Delta_j(a^*) \neq 0$ for every $a^* \in \mathcal{W}_\alpha$. Denote by Ξ_{j,a^*} the family of roots from $\overline{k}$ of the separable polynomial $H_j(a^*, Z)$. We assume that the sets of indices $J_{\alpha,s,r}$ are pairwise disjoint.

(vi) There is a family of polynomials $\Phi_{\alpha,s,j} \in k[a_1, \ldots, a_\nu, Z, Y_0, \ldots, Y_{s+1}]$, $j \in J_{\alpha,s,r}$, and polynomials $\lambda_0, \lambda_1 \in k[a_1, \ldots, a_\nu]$ (they depend on α, s, r) satisfying the following properties. For every $a^* \in \mathcal{W}_\alpha$ the polynomials $\Phi_{\alpha,s,j}$ are homogeneous with respect to $Y_0, \ldots, Y_{s+1}$, the degree $\deg_Z \Phi_{\alpha,s,j} < \deg_Z H_j$, the leading coefficient $\mathrm{lc}_{Y_{s+1}} \Phi_{\alpha,s,j} \in k[a_1, \ldots, a_\nu]$, all the polynomials $\Phi_{\alpha,s,j}(a^*, \xi, Y_0, \ldots, Y_{s+1})$, $\xi \in \Xi_{j,a^*}$, $j \in J_{\alpha,s,r}$, are irreducible over $\overline{k}$ in the ring $\overline{k}[X_0, \ldots, X_n]$, $\lambda_0(a^*) \neq 0$, $\lambda_1(a^*) \neq 0$ and

$$\Phi_{\alpha,s,r}(a^*, Y_0, \ldots, Y_{s+1}) = \frac{\lambda_0(a^*)}{\lambda_1(a^*)} \prod_{j \in J_{\alpha,s,r},\, \xi \in \Xi_{j,a^*}} \Phi_{\alpha,s,j}(a^*, \xi, Y_0, \ldots, Y_{s+1}).$$

Hence $\deg_{Y_0,\ldots,Y_{s+1}} \Phi_{\alpha,s,j} \leqslant \deg_{Y_0,\ldots,Y_{s+1}} \Phi_{\alpha,s,r} \leqslant d^{n-s}/p^r$.

(vii) For every $a^* \in \mathcal{W}_\alpha$ the irreducible over $\overline{k}$ components of the projective algebraic variety $V_{a^*,s}$ are in the natural one–to–one correspondence with pairs (ξ, j) where $\xi \in \Xi_{j,a^*}$, $j \in J_{\alpha,s,r}$, $0 \leqslant r \leqslant \rho$. Denote by $W_{j,a^*,\xi}$ the irreducible over $\overline{k}$ component of the algebraic variety $V_{a^*,s}$ corresponding to the pair (ξ, j). We have $\deg W_{j,a^*,\xi} = \deg_{Y_{s+1}} \Phi_{\alpha,s,j}$.

(viii) Let Y and Z be variables, $t_1, \ldots, t_s$ be a family of algebraically independent elements over $\overline{k}$, the element $j \in J_{\alpha,s,r}$ and $\Phi_{\alpha,s,j}(a^*, \xi, 1, t_1^{p^r}, \ldots, t_s^{p^r}, \theta^{p^r}) = 0$. Then there are polynomials $D_j \in k[a_1, \ldots, a_\nu, t_1, \ldots, t_s]$, $D_{j,i} \in k[a_1, \ldots, a_\nu, Z, t_1, \ldots, t_s, Y]$, $0 \leqslant i \leqslant n$, satisfying the following properties. The polynomial $D_j(a^*, t_1, \ldots, t_s) \neq 0$ for every $a^* \in \mathcal{W}_\alpha$, the degrees $\deg_Z D_{j,i} < \deg_Z H_j$, $\deg_Y D_{j,i} < \deg_{Y_{s+1}} \Phi_{\alpha,s,j}$ and all $\deg_{t_1,\ldots,t_s} D_j$, $\deg_{t_1,\ldots,t_s} D_{j,i}$ are bounded from above by $d^{O(n-s)}$. Further there is a $\overline{k}$-isomorphism of fields $\overline{k}(W_{j,a^*,\xi}) \to \overline{k}(t_1, \ldots, t_s)[\theta]$ such that $Y_i/Y_0 \mapsto t_i$, $1 \leqslant i \leqslant s$, $Y_{s+1}/Y_0 \mapsto \theta$,

$$(X_i/Y_0)^{p^r} \mapsto D_{j,i}(a^*, \xi, t_1^{p^r}, \ldots, t_s^{p^r}, \theta^{p^r})/D_j(a^*, t_1^{p^r}, \ldots, t_s^{p^r}), \quad 0 \leqslant i \leqslant n.$$

Hence this isomorphism gives a generic point of the algebraic variety $W_{j,a^*,\xi}$.

(ix) There is a finite family of linear forms $Y_{i,v} \in k[X_0, \ldots, X_n]$, $i \in I_{\alpha,s}$, $0 \leqslant v \leqslant s+2$, depending only on α, s and satisfying the following properties. For every $i \in I_{\alpha,s}$ for every $0 \leqslant v \leqslant s$ the linear form $Y_{i,v} = Y_v$. Further for every $j \in J_{\alpha,s,r}$, $0 \leqslant r \leqslant \rho$, $i \in I_{\alpha,s}$ there is a family of polynomials $\Psi_{\alpha,s,j,i,w} \in k[a_1, \ldots, a_\nu, Z, Y_{i,0}, \ldots, Y_{i,s+2}]$, $w \in I_{\alpha,s,j,i}$, homogeneous with respect to $Y_{i,0}, \ldots, Y_{i,s+2}$ and satisfying the following conditions. The degrees $\deg_Z \Psi_{\alpha,s,j,i,w} < \deg_Z H_j$, $\deg_{Y_{i,0},\ldots,Y_{i,s+2}} \Psi_{\alpha,s,j,i,w} \leqslant \deg_{Y_{s+1}} \Phi_{\alpha,s,j}$, finally for every point $a^* \in \mathcal{W}_\alpha$ the projective algebraic variety

$$W_{j,a^*,\xi} = \mathcal{Z}(\Psi_{\alpha,s,j,i,w}(a^*, \xi, Y_{i,0}^{p^r}, \ldots, Y_{i,s+2}^{p^r}),\ w \in I_{\alpha,s,j,i},\ i \in I_{\alpha,s}) \quad (3)$$

in $\mathbb{P}^n(\overline{k})$ and the number of elements $\#I_{\alpha,s} = d^{O(n-s)}$, $\#I_{\alpha,s,j,i} \leqslant d^{n-s}/p^r$ for all α, s, j, i. So (3) gives a system of homogeneous polynomial equations determining the algebraic variety $W_{j,a^*,\xi}$.

Now we able to formulate our main result.

THEOREM 1 *Let $f_0, \dots, f_{m-1} \in k[a_1, \dots, a_\nu, X_1, \dots, X_n]$ and $\mathcal{U}_c$ be as above. Then there is a stratification (2) satisfying the properties (i)–(ix) and such that*

(a) the number of elements $\#A$ and all the integers μ_α, $m_{\alpha,\beta}$ are bounded from above by $(d')^\nu d^{O(n\nu)}$ with an absolute constant in $O(n\nu)$,

(b) the degrees with respect to $a_1, \dots, a_\nu$ of all the polynomials $\psi^{(\beta)}_{\alpha,1}, \dots, \psi^{(\beta)}_{\alpha,m_{\alpha,\beta}}$, $\Phi_{\alpha,s,r}$, H_j, $\Phi_{\alpha,s,j}$, λ_0, λ_1, D_j, $D_{j,i}$, $\Psi_{\alpha,s,j,i,w}$ are bounded from above by $d' d^{O(n)}$ with an absolute constant in $O(n)$.

The proof of this theorem is based on the algorithm from [2] with some modifications and the results of [5], [6], see also [3], [4]. For the considered problem previously known estimates for degrees were double-exponential, cf. [1], [7].

References

[1] **Ayad A.:** *"Complexity of solving parametric polynomial systems"*, Zap. Nauchn. Semin. POMI v.387 (2011), p. 5–52

[2] **Chistov A. L.:** *"Polynomial complexity algorithm for factoring polynomials and constructing components of a variety in subexponential time"*, Zap. Nauchn. Semin. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 137 (1984), p. 124–188 (in Russian) [English transl.: J. Sov. Math. 34, (4), (1986) p. 1838–1882].

[3] **Chistov A. L.:** *"An improvement of the complexity bound for solving systems of polynomial equations"*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) 390 (2011), p. 299–306.

[4] **Chistov A. L.:** *"A Bound for the Degree of a System of Equations Determining the Variety of Reducible Polynomials"*, Algebra i Analiz 24 #3 (2012) p. 199–222 (in Russian), and *"Correction..."*, Algebra i Analiz 25 #2 (2013) p. 279 (in Russian) [English transl.: St. Petersburg Math. J. 24 #3 (2013), p. 513–528]

[5] **Chistov A. L.:** *"Computations with parameters: a theoretical background"*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) 436 (2015), p. 219–239 (in Russian) [English transl.: Journal of Mathematical Sciences 2016, Volume 215, Issue 6, p. 769–781].

[6] **Chistov A. L.:** *"Efficient absolute factoring polynomials with parametric coefficients"*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) 448 (2016), p. 286–325 (in Russian) [English transl.: to appear in St. Petersburg Math. J.]

[7] **Lazard D., Rouillier F.:** *"Solving parametric polynomial systems"*, Journal of Symbolic Computation v.42 #6 (2007), p. 636–667.

The greedy sequences of Young diagrams produced by Markov processes on 2D and 3D Young graphs

Vasilii Duzhin and Nikolay Vasilyev

There are many properties important for asymptotic combinatorics and asymptotic representation theory which can be described by special Markov processes on such graded graphs. In particular, there is the Plancherel process [1] where probabilities of diagrams are directly proportional to the square of the number of paths to the diagram, i.e. the dimension of the diagram.

We present the results of computer investigations of special so-called greedy sequences of two- and three-dimensional Young diagrams. Such sequences correspond to Markov processes on graded graphs and present infinite paths with maximum transition probability for each step. Our papers [2, 3, 4] were devoted to studying these sequences for Plancherel Markov processes on two-dimensional Young and Schur graphs.

In this work we continue this investigation and generalize it to the case of Markov processes on three-dimensional Young graph. We discuss the probability distribution on the limit front of diagrams from the greedy sequence. This distribution has an interesting resonant structure. We also consider some problems related to the investigation of limit shape of diagrams from the greedy seqence discussed in [5].

References

[1] A. M. Vershik and S. V. Kerov, "Asymptotic behavior of the maximum and generic dimensions of irreducible representations of the symmetric group.", Funktsional. Anal. i Prilozhen., 19(1):25-36, 1985.

[2] Vasilyev N. N., Duzhin V. S., "Building Irreducible Representations of a Symmetric Group S(n) with Large and Maximum Dimensions.", Informatsionno-upravliaiushchie sistemy [Information and Control Systems], 2015, no. 3, pp. 1722 (In Russian). doi:10.15217/issn1684-8853.2015.1.17

This work was supported by grant RFBR 17-01-00433.

[3] N. N. Vasilyev and V. S. Duzhin, "A study of the growth of maximal and typical normalized dimensions of strict Young diagrams.", J. Math. Sci. 216 (2016) 5364, doi: [doi:10.1007/s10958-016-2887-x]

[4] V. S. Duzhin and N. N. Vasilyev, "Asymptotic behavior of normalized dimensions of standard and strict Young diagrams growth and oscillations.", J. Knot Theory Ramifications 25, 1642002 (2016) [16 pages] DOI: http://dx.doi.org/10.1142/S0218216516420025

[5] N. N. Vasiliev, V. S. Duzhin, "Numerical investigation of the asymptotics of the probabilities of paths in a Markov process on the 3D Young graph close to a central one.", *Representation theory, dynamical systems, combinatorial and algoritmic methods. Part XXVII*, Zap. Nauchn. Sem. POMI, 448, POMI, St. Petersburg, 2016, 6979

Vasilii Duzhin
Saint Petersburg Electrotechnical University
ul. Professora Popova 5, 197376 St. Petersburg, Russian Federation
e-mail: vduzhin.science@gmail.com

Nikolay Vasilyev
St.Petersburg department of Steklov Institute of mathematics RAS
nab.Fontanki 27,St.Petersburg, 191023, Russia
e-mail: vasiliev@pdmi.ras.ru

Finding New Integrals of the Algaba System

Victor Edneral

Abstract. We consider an autonomous system of ordinary differential equations, which is resolved with respect to derivatives. To study local integrability of the system near a degenerate stationary point, we use an approach based on Power Geometry[2] and on the computation of the resonant normal form[3, 4]. For the partial non Hamilton 5-parameter case of concrete planar system[1], we found the complete set of necessary conditions on parameters of the system for which the system is locally integrable near a degenerate stationary point. These sets of parameters, satisfying the conditions, consist of 4 two-parameter subsets in this 5-parameter space. The first integral of motion corresponds of each such subset[5]. But along the hyper plane $b^2 = 2/3$ there can exist additional such subsets[6]. We have found two more first integrals of motion.

Acknowledgements

This work was supported by grant NSh-7989.2016.2 of the President of Russian Federation and by the Ministry of Education and Science of the Russian Federation (the Agreement number 02.a03.21.0008).

References

[1] A. Algaba, E. Gamero, C. Garcia, The integrability problem for a class of planar systems, Nonlinearity **22** (2009) 395–420

[2] A.D. Bruno, Power Geometry in Algebraic and Differential Equations, Fizmatlit, Moscow, 1998 (Russian) = Elsevier Science, Amsterdam, 2000 (English)

[3] A.D. Bruno, Local Methods in Nonlinear Differential Equations, Nauka, Moscow, 1979 (Russian) = Springer-Verlag, Berlin, 1989 (English)

[4] V.F. Edneral, On algorithm of the normal form building, in: Ganzha et al. (Eds.) Proceedings of the CASC 2007, Springer-Verlag series: LNCS 4770 (2007) 134–142

[5] V. F. Edneral and V. G. Romanovsky, Calculation of First Integrals of a Two Dimensional ODE System near a Degenerate Stationary Point by Computer Algebra Tools, Programming and Computer Software, **37** 2 (2011) 99–103

[6] Alexander D. Bruno, Victor F. Edneral, Possibility of Additional Solutions of the Degenerate System Near Double Degeneration at the Special Value of the Parameter, in V.P.Gerdt et.al. (Eds.) Proceedings the CASC 2013, Springer-Verlag series: LNCS 8136 (2013) 75–87

Victor Edneral
Skobeltsyn Institute of Nuclear Physics,
Lomonosov Moscow State University
Leninskie Gory 1(2), Moscow, 119991, Russian Federation

Peoples' Friendship University of Russia (RUDN University)
6 Miklukho-Maklaya st., Moscow, 117198, Russian Federation

e-mail: edneral@theory.sinp.msu.ru
e-mail: edneral_vf@pfur.ru

On Matveev's question about virtual 3-manifolds

E. A. Fominykh, A. V. Malyutin, and A. Yu. Vesnin

Abstract. S. V. Matveev introduced the concept of virtual 3-manifolds and posed a problem, whether the natural map from the set of nondegenerate virtual 3-manifolds to the set of 3-manifolds with RP^2-singularities is injective. We show that this map is not injective and present an infinite family of virtual 3-manifolds corresponding to the same 3-manifold with RP^2-singularities.

Matveev's virtual 3-manifolds can be naturally defined either in terms of *spines* or in the 'dual' language of singular triangulations. Contrary to the original definition via spines, we start with triangulations.

Face identification schemes. Let n be a positive integer, let $\mathcal{D} = \{\Delta_1, \dots, \Delta_n\}$ be a set of n disjoint tetrahedra, and let $\Phi = (\phi_1, \dots, \phi_{2n})$ be a collection of $2n$ affine homeomorphisms between the facets (i. e., triangular faces) of tetrahedra in $\mathcal{D}$ such that each facet has a unique counterpart. Following [Mat03, p. 11], we refer to the pair $(\mathcal{D}, \Phi)$ as to a *face identification scheme* (a *scheme*). The *quotient space* $Q := Q(\mathcal{D}, \Phi)$ of the scheme $(\mathcal{D}, \Phi)$ is defined as the space obtained from $\mathcal{D}$ by identification of faces via the homeomorphisms in Φ. For any scheme, Q is either a genuine or a singular 3-manifold (see [Sei33] or [Mat03, Proposition 1.1.23]). All singular points of Q correspond either to vertices or to barycenters of edges of the tetrahedra in $\mathcal{D}$. The latter happens if the quotient map folds an edge so that symmetric points (with respect to the barycenter of the edge) have the same image. If a singularity point x in Q corresponds to the barycenter of an edge, then the link of x is himeomorphic to the *projective plane* RP^2, i. e., x is an RP^2-singularity.

Pseudo-Pachner move. In Euclidean three-space, let P be the convex hull of five points that are in general position. Assume that P is not a tetrahedron. Then P is the union of two geometric tetrahedra (Δ_1 and Δ_2, say) and at the same time P is the union of three geometric tetrahedra with disjoint interiors (Δ_3, Δ_4, and Δ_5, say). Let

$$\tau = \partial\Delta_1 \cup \partial\Delta_2 \qquad \text{and} \qquad \tau' = \partial\Delta_3 \cup \partial\Delta_4 \cup \partial\Delta_5.$$

Supported by RFBR grant 16-01-00609.

If $Q := Q(\mathcal{D}, \Phi)$ is the quotient space of the scheme $(\mathcal{D}, \Phi)$, we denote by $\sigma(Q)$ the image in Q of boundaries of tetrahedra in $\mathcal{D}$ (*2-skeleton* of $(\mathcal{D}, \Phi)$).

We say that two schemes $(\mathcal{D}, \Phi)$ and $(\mathcal{D}', \Phi')$ are *related by a pseudo-Pachner move* if there exists a homeomorphism $h\colon Q \to Q'$ between the quotient spaces $Q := Q(\mathcal{D}, \Phi)$ and $Q' := Q(\mathcal{D}', \Phi')$ and a map $f\colon P \to Q'$ such that

– the restriction of f to the interior of P is an embedding;

– $h(\sigma(Q)) \cap f(P) = f(\tau)$ and $\sigma(Q') \cap f(P) = f(\tau')$;

– $h(\sigma(Q)) \setminus f(P) = \sigma(Q') \setminus f(P)$, that is, $h(\sigma(Q))$ and $\sigma(Q')$ do coincide outside $f(P)$.

Definition. Virtual 3-manifolds. We say that two face identification schemes are *equivalent* if they are related by a chain of pseudo-Pachner moves. A *virtual 3-manifold* is an equivalence class of schemes. We say that a virtual 3-manifold is *degenerate* if it corresponds to a scheme with precisely one tetrahedron.

Truncated quotient spaces as underlying spaces of virtual 3-manifolds. The *truncated quotient space* $Q_t := Q_t(\mathcal{D}, \Phi)$ of the scheme $(\mathcal{D}, \Phi)$ is defined as the space obtained from the quotient space $Q := Q(\mathcal{D}, \Phi)$ by deleting small open proper neighborhoods of points in Q that are images of vertices of the tetrahedra in $\mathcal{D}$. Truncated quotient spaces of equivalent schemes are obviously homeomorphic. By the *underlying space* $Q_t(V)$ of a virtual 3-manifold V we will mean the truncated quotient space of schemes in V. Thus, the underlying space of a virtual 3-manifold is a 3-manifold with RP^2-singularities.[1] The following facts follow from key results of the *special spine theory* (see [Mat03, Mat09]).

Corollary 1. (1) *Each compact connected 3-manifold M with RP^2-singularities and nonempty boundary is the underlying space of a virtual 3-manifold.*

(2) *If M is, moreover, genuine, then it is the underlying space of a unique nondegenerate virtual 3-manifold.*

Problem. The following natural question arises (see [Mat09]). Do the second statements of Corollary 1 holds for 3-manifolds with RP^2-singularities? That is, are there distinct nondegenerate virtual 3-manifolds with the same connected underlying space?

We present the following theorem that answers the question above.

Theorem 1. *The cone* $\mathrm{C}(RP^2) = (RP^2 \times [0,1])/(RP^2 \times \{0\})$ *over the projective plane RP^2 is the underlying space for an infinite number of pairwise distinct virtual 3-manifolds.*

In order to explain the idea of the proof of Theorem 1, we introduce the concepts of *3-manifolds with traced RP^2-singularities*.

[1] A compact three-dimensional polyhedron W is called a *3-manifold with RP^2-singularities* if the link of any point of W is either a 2-sphere, or a 2-disc, or RP^2.

3-manifolds with traced RP^2-singularities. A *3-manifold with traced RP^2-singularities* is a pair $(W, \mathcal{I})$, where W is a 3-manifold with RP^2-singularities and $\mathcal{I}$ is a subpolyhedron of W with the following properties:

– $\mathcal{I}$ is a disjoint union of arcs (these arcs are called *traces*),

– the number of components in $\mathcal{I}$ equals the number of singular points in W,

– each component of $\mathcal{I}$ emanates from the boundary of W and ends at a singularity point of W.

We describe two interpretations for 3-manifolds with traced RP^2-singularities.

3-manifolds with Möbius singularities. We say that a compact 3-dimensional polyhedron W is a *3-manifold with Möbius singularities* if the link of any point of W is either a 2-sphere, or a 2-disc, or a Möbius band. (The set of points of W whose links are 2-discs or Möbius bands form the *boundary* ∂W of W, so that the singularities of W are contained in its boundary.) Collapsing traces to points, we see that the set of 3-manifold with traced RP^2-singularities is in a natural one-to-one correspondence with the set of 3-manifolds with Möbius singularities.

3-manifolds decorated with orientation-reversing curves. A closed simple curve γ on a surface is *orientation-reversing* if a tubular neighbourhood of γ is a Möbius band. If W is a genuine 3-manifold and C is a collection of closed simple pairwise disjoint orientation-reversing curves on the boundary ∂W, then collapsing each curve in C to a point transforms W to a 3-manifolds with Möbius singularities (each curve of C gives a singularity point). Thus, the set of 3-manifolds equipped with orientation-reversing curves is another interpretation for the set of 3-manifolds with traced RP^2-singularities.

From virtual 3-manifolds to 3-manifolds with traced RP^2-singularities. Each virtual 3-manifold is naturally assigned with a 3-manifold with traced RP^2-singularities. Indeed, observe that all singular points of the truncated quotient space $Q_t := Q_t(\mathcal{D}, \Phi)$ of the scheme $(\mathcal{D}, \Phi)$ are RP^2-singular points corresponding to barycenters of edges of the tetrahedra of $\mathcal{D}$. The quotient map folds the edges that give singular points so that symmetric points (with respect to the barycenters) of each such *folding* edge have the same image. We observe that the pair $(Q_t, \mathcal{I})$, where $\mathcal{I}$ is the image in Q_t of all folding edges, is a 3-manifold with traced RP^2-singularities. The described correspondence $(\mathcal{D}, \Phi) \mapsto (Q_t, \mathcal{I})$ associates a 3-manifold with traced RP^2-singularities to each scheme. Now, observe that any pseudo-Pachner move changes neither Q_t nor the subset in Q_t formed by the images of folding edges, so that equivalent schemes yield the same 3-manifold with traced RP^2-singularities. This gives a map (which we denote by F) from the set of virtual 3-manifolds to the set of 3-manifolds with traced RP^2-singularities. Note that by construction, forgetting the traces in $F(V)$, where V is a virtual 3-manifold, converts $F(V)$ to the underlying space of V.

Theorem 2. *If $(M, \mathcal{I})$ is a compact connected 3-manifold with traced RP^2-singularities and nonempty boundary, then there exists a virtual 3-manifold V such that $F(V) = (M, \mathcal{I})$.*

Lemma 1. *The cone* $\mathrm{C}(RP^2) = (RP^2 \times [0,1])/(RP^2 \times \{0\})$ *over the projective plane* RP^2 *is the base space for an infinite number of pairwise distinct* 3*-manifolds with traced* RP^2*-singularities. In other words, the singularity point in* $\mathrm{C}(RP^2)$ *can be traced in an infinite number of pairwise nonequivalent ways.*

Ideas of proofs. Theorem 1 follows from Theorem 2 and Lemma 1.

Theorem 2 can be proved in terms of special spine theory on the base of results established in [Mat03]. Given a compact connected 3-manifold with traced RP^2-singularities $(M, \mathcal{I})$, we delete from M open proper neighbourhoods of $\mathcal{I}$. The obtained genuine 3-manifold $M_{\mathcal{I}} \subset M$ possesses a *special spine* S (see [Cas65] or [Mat03, Theorem 1.1.13]). Observe that $\partial M_{\mathcal{I}} \setminus \partial M$ is the union of disjoint open Möbius bands. Let $C \subset \partial M_{\mathcal{I}}$ be the union of center curves of these Möbius bands. A natural retraction $p\colon M_{\mathcal{I}} \to S$ sends C to a collection of curves on S. We attach to S new *unthickenable* 2-cells along $p(C)$ and show that the obtained special polyhedron represents a virtual 3-manifold V with $F(V) = (M, \mathcal{I})$.

In order to prove Lemma 1, consider 'knotted' traces and use the branched double covering of $\mathrm{C}(RP^2)$ by the 3-ball (ramified over the singularity point). □

References

[Cas65] B. G. Casler, *An imbedding theorem for connected 3-manifolds with boundary*, Proc. Amer. Math. Soc., **16**, 559–566 (1965).

[VTF16] A. Yu. Vesnin, V. G. Turaev, E. A. Fominykh, *Complexity of virtual 3-manifolds*, Mat. Sb., **207**:11 (2016), 4–24; Sb. Math., **207**:11 (2016), 1493–1511.

[Mat03] S. V. Matveev, *Algorithmic topology and classification of 3-manifolds*, Algorithms and Computation in Mathematics, vol. 9, Springer-Verlag, Berlin, 2003.

[Mat09] S. V. Matveev, *Virtual 3-manifolds*, Siberian electronic mathematical reports **6** (2009), 518–521.

[Sei33] H. Seifert, *Topologie Dreidimensionaler Gefaserter Räume*, Acta Math., **60** (1933), 147–238.

E. A. Fominykh
Chelyabinsk State University and Krasovskii Institute of Mathematics and Mechanics
Chelyabinsk, Russia
e-mail: efominykh@gmail.com

A. V. Malyutin
St. Petersburg Department of Steklov Institute of Mathematics
St. Petersburg, Russia
e-mail: malyutin@pdmi.ras.ru

A. Yu. Vesnin
Sobolev Institute of Mathematics
Novosibirsk, Russia
e-mail: vesnin@math.nsc.ru

On computer modeling of finite-generated free projective planes

Gogin Nikita and Mylläri Aleksandr

Abstract. This paper is devoted to computer modeling of the process of constructing free projective planes - more precisely, to algorithmically finding their successive incidence matrices - and also to considering some numerical characteristics of these matrices.

Introduction

Free projective planes were first introduced by M. Hall in his fundamental paper [1] where he considered their basic properties. Since then, these planes have become the subject of constant interest of mathematicians studying abstract algebraic structures, group theory and their representations, and so on [2, 3, 5, 6, 7]. There are also good surveys which one can use to get acquainted with the basic concepts and achievements of the modern theory of combinatorial geometries, for example, [4, 8].

This paper is devoted to computer modeling of the process of constructing free projective planes - more precisely, to algorithmically finding their successive incidence matrices - and also to considering some numerical characteristics of these matrices.

Remarks about notations: If A is a (non-empty) matrix then $dim1(A)$ (resp. $dim2(A)$) is a number of its rows (resp. columns); $[A]_{i,j}$ means its element at the entry (i, j); A_i (resp. A^j) means $i-$th row (resp. j-th column); $diag(A)$ for a square matrix A means column-vector of its diagonal elements; $Total[A]$ is a sum of all elements in A. Moreover, we treat binomial coefficient $\binom{x}{2}$ and differential operators (derivatives, Laplace operator) as listable functions.

$\eta_{i,j}$ is a column-vector with "1"$-$s only in two different positions i and j and all the rest components equal to "0"$-$s.

As a rule we do not show the matrix format explicitly until it is not clear from context.

E denotes identity matrix; J is a square constant matrix of (only) "1"−s; $J^* = J - E$; $\{\}$ denotes empty matrix; $\langle\ ,\ \rangle$ means Euclidean scalar product; for a matrix A and real α we define a product $\alpha \bullet A$ as follows: $\alpha \bullet A = \begin{cases} \alpha A, & \text{if } \alpha \neq 0 \\ \{\}, & \text{if } \alpha = 0. \end{cases}$

$A \circ B$ denotes the element-wise (Hadamard) product of matrices with the identical formats.

If A and B are matrices having appropriate formats then $A| \cup B$(resp. $\underline{A} \cup B$) denotes a concatenation of A and B from the right (resp. "from below") providing $A| \cup \{\} = \underline{A} \cup \{\} = A$.

1. Preliminaries

In this section we mostly follow the terminology and definitions of book [4].

Definition 1. A configuration (or a partial plane [1]) is a pair $\Pi = (P, L)$ where P is (nonempty) set of points and L is a family of subsets of P called lines under the condition that the following axiom is valid:

C1: Any two different points are incident with no more than one line.

Axiom C1 implies

C2: Any two different lines are incident with no more than one point in common. [4]

As a rule in this paper we shall be interested only in the case of finite sets P.

Examples 1.

1. Desargues and Pappus configurations are well-known (cf. [4]) .

2. If in Definition 1 $L = \emptyset$ and $|P| = m$, $m > 0$ is an integer, then we have a pure $m-$points configuration.

3. If L consists of all pairs $\{a, b\}$, $a, b \in P$, $a \neq b$ then $\Pi = (P, L)$ is a full graph on m vertices.

4. Let $\Pi^m = (P, L)$, $m \geq 4$ be a configuration with $|P| = m$ and only one line λ, (i.e. $L = \{\lambda\}$) where $|\lambda| = m - 2$. This means that all points besides two of them lie on the (unique) line λ. These configurations are called standard [8] or Hall configurations and were first introduced by M. Hall in his fundamental paper [1], p. 237.

Definition 2. Configuration $\Pi = (P, L)$ is called a projective plane, if in axioms C1 and C2 the words "...with no more than one..." are changed by "... exactly one...", i.e. in $\Pi = (P, L)$ the following axioms are valid:

P1: Any two different points are incident to exactly one line;

P2: Any two different lines are incident to exactly one point in common;

and in addition the axiom

P3: There exist 4 different points such that no three of them are collinear; in order to exclude some degenerate configurations (cf. [8]).

The following simple statements can be easily proved for a finite projective plane [2]:

A) Every line is incident to exactly $n + 1$ points;

B) Every point is incident to exactly $n+1$ lines;
C) $|P| = |L| = N = n^2 + n + 1$.
The number n is called the order of the finite projective plane.

"Prime-power hypothesis for the orders of the finite projective planes" claims that always $n = p^\mu$ for some prime p. Nowadays this hypothesis remains unproved.

If $\Pi = (P, L)$ is a finite configuration with $|P| = m$ and $|L| = l$, $l > 0$ then the incident matrix of Π is defined as $l \times m$ 0-1-matrix $A = (a_{i,j})$ where

$$a_{i,j} = \begin{cases} 1, & \text{if point } j \text{ is incident with line } i \\ 0, & \text{if point } j \text{ and the line } i \text{ are not incident} \end{cases} \quad 1 \le i \le L.1 \le j \le m \qquad (1)$$

in some chosen (and fixed) numerations of sets P and L.

Obviously

$$\begin{aligned} Total[A_i] &= \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{i,j}^2 \qquad (2) \\ &= \langle A_i, A_i \rangle = (\text{number of points on the } i-\text{th line}) \end{aligned}$$

$$\begin{aligned} Total[A^j] &= \sum_{i=1}^{l} a_{i,j} = \sum_{i=1}^{l} a_{i,j}^2 \qquad (3) \\ &= \langle A^j, A^j \rangle = (\text{number of lines incedent to the } j-\text{th point}) \end{aligned}$$

$$Tr(AA^T) = Tr(A^T A) = Total(A) \qquad (4)$$

If all the outside-diagonal elements in AA^T (resp., $A^T A$) are equal to 1, we say that configuration is line-wise ample (resp. point-wise ample).

Clearly, if $\Pi = (P, L)$ is a projective plane of order n then it is both point-wise ample and line-wise ample and its incident matrix is a square $N \times N$ 0-1-matrix such that

$$AA^T = A^T A = nE + J \qquad (5)$$

(cf. for example, [2]).

2. Free projective plane generated by configuration

Let $\Pi_0 = (P_0, L_0)$ be some (initial) configuration. The free projective plane generated by Π_0 is defined by the following process:
1. Let $\Pi_1 = (P_1, L_1)$ be a new configuration where $L_1 = L_0$ and $P_1 = P_0 \cup \nu P_0$

$$\nu P_0 = \{(a)(b) | a, b \in L_0, a \text{ and } b \text{ are not incident in } \Pi_0\} \qquad (6)$$

i.e. every pair of non-incident lines defines a new point named $(a)(b)$ which is "intersection" of lines a and b. Evidently Π_1 is line-wise ample.

2. Let $\Pi_2 = (P_2, L_2)$ be a new configuration where $P_2 = P_1$ and $L_2 = L_1 \cup \nu L_1$

$$\nu L_1 = \{(a)(b) | a, b \in P_1, a \text{ and } b \text{ are not incident in } \Pi_1\} \tag{7}$$

i.e. every pair of non-incident points a and b defines a new line named $(a)(b)$ which "connects" points a and b. Evidently Π_2 is point-wise ample.

Iterating this construction we get a sequence (finite or infinite) of configurations $\{\Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5, \dots, \Pi_r, \dots\}$ in which for r even we add new points to Π_r, as in item 1 and for r odd we add new lines to Π_r as in item 2 and get next configuration Π_{r+1}, $r \geq 0$.

Proposition 1 (see [4]). If Π_0 contains 4 different points no three of which are collinear then $\Pi = fr(\Pi_0) = \bigcup_{k=0}^{\infty} \Pi_k$ is a projective plane.

This plane is said to be the free projective plane generated by Π_0.

Example 2

1. If Π_0 is a projective plane then evidently $fr(\Pi_0) = \Pi_0$.

2. If $|\Pi_0| = 3$ and $|L_0| = 0$ then $fr(\Pi_0)$ is called a "projective plane of order $n = 1$" (see Definition 2, p.1) and it is a plane over the field of one element (Fig.1, left). Its incident matrix is cyclic.

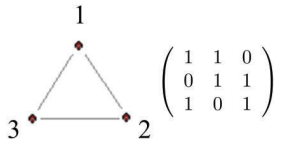


FIGURE 1. Projective plane of order $n = 1$ (left) and its incident matrix (right).

The following theorem of M. Hall (see [1]) explains the importance of Hall configuration Π^4 :

1) Let Π_0 is any non-degenerate configuration but not a projective plane. Then $fr(\Pi_0)$ contains $fr(\Pi^4)$ as a subplane. Moreover, such plane is never desarguesian.

2) A $fr(\Pi^m)$, $m \geq 4$ contains $fr(\Pi^{m+1})$.

Everywhere in what follows we deal only with the Hall configuration Π^4, i.e. $fr(\Pi^4) = \{\Pi^4_r\}_{r=0,1,2,\dots}$, that is "free equivalent"(see [1]) to pure configuration on 4 points, or full graph with 4 vertices.

3. Matrix approach

According to what was said at the end of previous section we begin with configuration $\Pi_0 = \Pi^4$ (which is zero-step, $s = 0$, of our algorithm) with incident

matrix

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

which corresponds to the configuration 3 from Example 1 with $m = 4$. This configuration (tetrahedron) is shown below on Fig. 2 (left), where the numeration of lines is omitted.

Evidently here $\dim 1(A_0) = \Lambda_0 = 6$, $\dim 2(A_0) = P_0 = 4$. Since

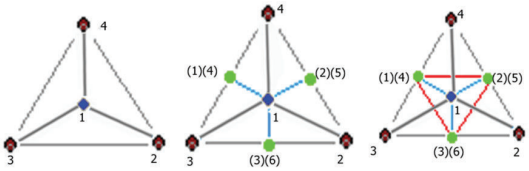


FIGURE 2. Initial configuration $\Pi_0 = \Pi^4$ (left) and two steps of the algorithm: adding new points (center) and new lines (right).

$$A_0 A_0^T = \begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \end{pmatrix} \quad A_0^T A_0 = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

this configuration is point-wise ample (any two different points are incident), but is not line-wise ample because exactly 3 pairs of lines, namely 1,4, 2,5 and 3,6, have no points in common.

According to item 1 of the general constructing of $fr(\Pi_0)$ at the next step $s = 1$ we must add to Π_0 $\nu P_0 = 3$ new points, namely (1)(4), (2)(5) and (3)(6) (see Fig. 2), that means that we must concatenate (from the right) to A_0 three new columns numbered respectively 5, 6, 7, whereas the amount of new lines $\nu A_0 = 0$.

So, here $\dim 1(A_1) = \Lambda_1 = \Lambda_0 = 6$, $\dim 2(A_1) = P_0 + \nu P_0 = 4 + 3 = 7$ and the matrix of the next configuration Π_1 (see Fig.2 (center)) is

$$A_1 = \left(\begin{array}{cccc|ccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Note that positions of "1"−s in the concatenated columns are (clearly why) exactly 1 and 4, 2 and 5, and 3 and 6.

Going over to the next step $s = 2$ we find that

$$A_1 A_1^T = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix} \quad A_1^T A_1 = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2 \end{pmatrix}$$

So, here $\dim 1(A_2) = \Lambda + 2 = \Lambda_1 + \nu\Lambda_1 = 6 + 3 = 9$, $\dim 2(A_2) = P_2 = P_1 + \nu P_1 = 7 + 0 = 7$ and the matrix of the next configuration (see Fig.2 center) is

$$A_2 = \left[\begin{array}{ccccccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Now it is not difficult to describe the general case for any step $s > 0$:

a1) If $s \equiv 1 \mod 2$ we add new points

$$\begin{aligned} \nu P_{s-1} &= \text{(number of non-incident lines at step } s-1) \\ &= \frac{1}{2}\,(\text{number of "0"-s in } A_{s-1}A_{s-1}^T) \\ &= \binom{\Lambda_{s-1}}{2} - Total\left[\binom{diag(A_{s-1}^T A_{s-1})}{2}\right] \\ &= \binom{\Lambda_{s-1}}{2} - Total\left[\binom{A_{s-1}^T A_{s-1}}{2}\right] \end{aligned} \tag{8}$$

whereas clearly $\nu A_{s-1} = 0$.

So, in this case we get a formula (we remind that $0 \bullet a = \{\}$):

$$A_s = A_{s-1} | \bigcup_{2 \le i \le A_{s-1}\, i \le j \le A_{s-1}} \left((1 - [A_{s-1}A_{s-1}^T]_{i,j}) \bullet \eta_{i,j}\right) \tag{9}$$

Dually,

a2) If $s \equiv 0 \mod 2$ we add new lines

$$\begin{aligned} \nu\Lambda_{s-1} &= (\text{number of non-incident points at step } s-1 \\ &= \frac{1}{2}\,(\text{number of "0"-s in } A_{s-1}^T A_{s-1}) \\ &= \binom{P_{s-1}}{2} - Total\left[\binom{diag(A_{s-1}A_{s-1}^T)}{2}\right] \\ &= \binom{P_{s-1}}{2} - Total\left[\binom{A_{s-1}A_{s-1}^T}{2}\right] \end{aligned} \tag{10}$$

whereas clearly $\nu P_{s-1} = 0$.

For example, for $s = 2$ we get $\nu\Lambda_1 = \binom{7}{2} - 6 \cdot \binom{3}{2}$, since $P_1 = 7$, $diag(A_1 A_1^T) = (3\,3\,3\,3\,3\,3\,3)$. So, in this case we get a formula:

$$A_s = \underline{A_{s-1}} \bigcup_{2 \le i \le A_{s-1} i \le j \le A_{s-1}} \left((1 - [A_{s-1}^T A_{s-1}]_{i,j}) \bullet \eta_{i,j}^T\right) \tag{11}$$

Formulas (9) and (11) give rise to the first variant of our algorithms.

4. Bilinear forms approach

Let $\pi = \{p_i\}_{i=1}^{\infty}$ and $\lambda = \{l_i\}_{i=1}^{\infty}$ be two sets of independent variables for points and lines respectively.

For any step $s \ge 0$ we introduce a bilinear form $F_s = F_s(\pi, \lambda) = \pi^T A_s \lambda$ where A_s is an incident matrix constructed on step s (see Sec. 3) and π and λ are initial segments of the infinite sequences of variables π and λ having appropriate lengths. For example, for $s = 0$ we have $\pi = \{p_i\}_{i=1}^{4}$, $\lambda = \{l_i\}_{i=1}^{6}$ and

$$\begin{aligned} F_0 &= l_4p_1 + l_5p_1 + l_6p_1 + l_2p_2 + l_3p_2 + l_4p_2 + l_1p_3 + l_3p_3 + l_5p_3 + l_1p_4 \\ &\quad + l_2p_4 + i_6p_4 \\ &= l_1(p_3 + p_4) + l_2(p_2 + p_4) + l_3(p_2 + p_3) + l_4(p_1 + p_2) + l_5(p_1 + p_3) \\ &\quad + l_6(p_1 + p_4) \\ &= p_1(l_4 + l_5 + l_6) + p_2(l_2 + l_3 + l_4) + p_3(l_1 + l_3 + l_5) + p_4(l_1 + l_2 + l_6) \end{aligned} \tag{12}$$

Now it is clear that also in general case $Coefficient[F_s, l_i] = \frac{\partial F_s}{\partial l_i}$ is a linear form in π representing the $i-$th row of A_s; $Coefficient[F_s, p_j] = \frac{\partial F_s}{\partial p_j}$ is a linear form in λ representing the $j-$th column of A_s.

Also it is clear that two lines, l_i and l_k with $1 \le i, k \le \Lambda_s$, $i \ne k$ are not incident iff. the linear forms $\frac{\partial F_s}{\partial l_i}$ and $\frac{\partial F_s}{\partial l_k}$ have no variables in common that implies that in this case the Laplace operator in π

$$\Delta_\pi \left(\frac{\partial F_s}{\partial l_i} \cdot \frac{\partial F_s}{\partial l_k}\right) = \sum_{p \in \pi} \frac{\partial^2}{\partial p^2} \left(\frac{\partial F_s}{\partial l_i} \cdot \frac{\partial F_s}{\partial l_k}\right) = 0 \tag{13}$$

and otherwise

$$\Delta_\pi \left(\frac{\partial F_s}{\partial l_i} \cdot \frac{\partial F_s}{\partial l_k} \right) = 2. \tag{14}$$

It's clear that if $i = k$ then

$$\lambda_\pi \left(\left(\frac{\partial F_s}{\partial l_i} \right)^2 \right) = 2 \cdot \text{ (number of points on } i - \text{th line)} = 2 \left[diag(A_s A_s^T) \right]_i \tag{15}$$

For example,

$$\Delta_\pi \left(\frac{\partial F_0}{\partial l_1} \cdot \frac{\partial F_0}{\partial l_4} \right) = \sum_{r=1}^{4} \frac{\partial^2}{\partial p_r^2} \left((p_3 + p_4)(p_1 + p_2) \right) = 0,$$

whereas

$$\Delta_\pi \left(\frac{\partial F_0}{\partial l_1} \cdot \frac{\partial F_0}{\partial l_2} \right) = \sum_{r=1}^{4} \frac{\partial^2}{\partial p_r^2} \left((p_3 + p_4)(p_2 + p_4) \right) = 2,$$

and

$$\Delta_\pi \left(\frac{\partial F_0}{\partial l_1} \right)^2 = \sum_{r=1}^{4} \frac{\partial^2}{\partial p_r^2} \left((p_3 + p_4)^2 \right) = 2 \cdot 2 = 4.$$

Obviously that formulas dual to (13), (14) and (15) also are valid mutatis mutandis.

Using formulas (13), (14), (15) and their duals it is easy to verify matrices equalities

$$\frac{1}{2} \Delta_\pi \left(\left(\frac{\partial F_s}{\partial \lambda} \right)^{\otimes 2} \right) = A_s A_s^T, \frac{1}{2} \Delta_\lambda \left(\left(\frac{\partial F_s}{\partial \pi} \right)^{\otimes 2} \right) = A_s^T A_s, \tag{16}$$

where $\frac{\partial F_s}{\partial \lambda} = grad_\lambda(f_s)$, $\frac{\partial F_s}{\partial \pi} = grad_\pi(f_s)$, the Laplace operators are supposed to be listable and $^{\otimes 2}$ means tensor square.

These formulas also give rise to alternative algorithm for recursive construction of $fr(\Pi^4)$.

5. Implementation

As was said above we used "matrix approach", and "bilinear forms approach".

The first difficulty in programming was caused by the requirement to avoid zero-columns/rows in incident matrices as well as "fictitious" variables in bilinear forms. This difficulty is surmounted with special procedures for numeration of new constructed columns/rows of matrices and new variables of bilinear forms.

Rather more serious obstacle is the (mentioned above) fact of the very fast growth of matrices' formats. Though those are very sparse 0-1-matrices, the programming tools for such matrices provided by Mathematica occurred to be not sufficient for our purposes, so, the computer memory resources became exhausted very soon...

So, we managed to calculate only 7 members of the sequence $u_n = \nu P_n + \nu \Lambda_n$, $n \geq 0$ (note that one of the two summand in "u_n" is always equal to 0):

3, 3, 6, 24, 282, 37233, 684792168, ...

It is easy to check empirically that this sequence grows asymptotically as a double exponent of n (Fig. 3).

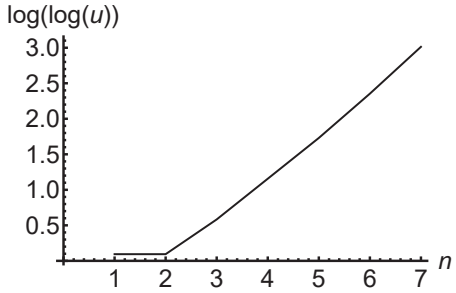


FIGURE 3. Number of elements grows as double exponent (linear on log(log) scale.

Authors thank Dr. Ian V. J. Murray, Dept Physiology and Neuroscience, St. George's University for the collaborative purchase of Wolfram *Mathematica*.

References

[1] M. Hall, *Protective planes*, Trans. Amer. Math. Soc., 54, 229-277, 1943.

[2] M. Hall, *The Theory of Groups*, NY, 1959.

[3] L. I. Kopejkina, *Decomposition of Protective Planes*, Bull. Acad. Sci,USSR, Ser. Math. Izvestia Akad. Nauk, SSSR, 9, 495-526, 1945.

[4] R. Hartshorne, *Protective Planes*, Lecture Notes Harvard University, NY, 1967.

[5] L.C. Siebenmann, *A Characterization of Free Projective Planes*, Pac. J. of Math., 15, 1, 293-298, 1965.

[6] R. Sandler, *The Collineation Groups of Free Planes*, Trans. Amer. Math. Soc., 107, 129-139, 1963.

[7] Ki Hang Kim, F.W. Roush, *A Universal Algebra Approach to Free Projective Planes*, Aequationes Mathematicae, University of Waterloo, 18, 389-400, 1978.

[8] A. I. Shirshov, A. A. Nikitin *Algrbraic Theory of Projective Planes*, Novosibirsk, 1987.

Gogin Nikita
Vyborg, Russia
e-mail: ngiri@list.ru

Mylläri Aleksandr
Dept. of Computers & Technology, SAS
St.George's University
St.George's, Grenada, West Indies
e-mail: amyllari@sgu.edu

Diagonal complexes

Joseph Gordon and Gaiane Panina

Assume that $n > 2$ is fixed. We say that two diagonals in a convex n-gon are *non-intersecting* if they intersect only at their endpoints (or do not intersect at all). John Milnor showed that the poset of all collections of pairwise non-intersecting diagonals in the n-gon (ordered by reverse inclusion) is isomorphic to the face poset of some convex $(n-3)$-dimensional polytope As_n called *associahedron*.

Instead of a polygon let us take an arbitrary (possibly bordered) orientable surface with a number of marked points (=vertices) lying not necessarily on the boundary. Generalizing a construction of J.L. Harer, we introduce and study similar diagonal complexes $\mathcal{C}$ and $\mathcal{B}$. Investigation of some natural forgetful maps combined with length assignment proves homotopy equivalence for some of the complexes, for the space of metric ribbon graphs $RG^{met}_{g,n}$, for the tautological S^1-bundles L_i, and for a more sophisticated bundle whose fibers are homeomorphic to some surgery of the surface F. The latter is shown to incorporate all the tautological S^1-bundles.

Acknowledgement

This research is supported by the Russian Science Foundation under grant 16-11-10039.

Joseph Gordon
Mathematics and Mechanics Faculty, St. Petersburg State University
e-mail: joseph-gordon@yandex.ru

Gaiane Panina
Mathematics and Mechanics Faculty, St. Petersburg State University, St. Petersburg Department of Steklov Institute of Mathematics
e-mail: gaiane-panina@rambler.ru

On the computational complexity of symmetric functions

Dima Grigoriev

Computing symmetric polynomials is a classical topic, in particular, in the complexity theory. Since this area is vast one considers usually, certain subclasses of symmetric polynomials.

We suppose to give a survey of earlier results and also to present two recent complexity bounds. The first one is an exponential lower bound (joint with G. Koshevoy) of a suitable monomial symmetric function by monotone calculations (so, using only additions and multiplications). The second one is an upper bound on the monotone complexity (joint with S. Fomin, D. Nogneng, E. Schost) of the complete symmetric polynomials, and as a consequence of Schur polynomials.

Also some open questions will be duscussed.

Dima Grigoriev
CNRS, France
Universite de Lille,
59655, Villeneuve d'Ascq, France

e-mail: dima.grigoryev@gmail.com

Efficient Calculation Managing on a Cluster with Distributed Memory

Evgeny Ilchenko and Gennadi Malaschonok

Abstract. Managing of cluster parallel computations for tree-like recursive algebraic algorithms for the case of cluster with distributed memory is one of the difficult problems of computer algebra. The block-recursive algorithms of matrix and polynomial multiplication, Strassen's and Karatsuba's algorithms, matrix inversion and computation of the kernel of a matrix operator, LDU and Bruhat factorization are examples of such algorithms. We suggest a scheme with multidispatching for management of such parallel computing processes and demonstrate the results of experiments at the JSC RAS cluster MVS-10P.

Introduction

The task of managing calculations on a cluster with distributed memory for sparse matrix algorithms is today one of the most difficult challenges [1]. This task can not be solved independently of the algorithm itself, since it can not be solved in an abstract setting. With any approach to such a task, it is necessary to somehow fix the class of algorithms.

We consider the class of block-recursive matrix algorithms. The most famous of them are standard and Strassen's block matrix multiplication, Schur and Strassen's block-matrix inversion [2].

For such algorithms we suggest a scheme with multidispatching management of parallel computing processes. We demonstrate the results of experiments at the cluster computer MVS-10P. These experiments show high efficiency of this conception of management.

Class of block-recursive matrix algorithms

Generalization of Strassen's algorithm for triangular factorization and matrix inversion with permutations of rows and columns by J. Bunch and J. Hopkroft [3] is not a block-recursive algorithm. Block-recursive algorithms were not so important

as long as the calculations were performed on computers with shared memory. Only in the nineties it became clear that block-recursive matrix algorithms are required to operate with sparse super large matrices on a supercomputer.

The block recursive algorithm for the solution of systems of linear equations and for adjoint matrix computation which is some generalisation of Schur inversion in commutative domains was discraibed in [4] and [5]. See also at the book [6]. However, in all algorithms, except matrix multiplication algorithms, there is a very strong restriction that is superimposed on the matrix. The leading minors of the matrix, which are on the main diagonal, should not be zero.

This restriction was removed later in the papers [7] - [9]. The algorithm that computes the adjoint matrix, the echelon form, and the kernel of the matrix operator for the commutative domains was proposed in [7]. The block-recursive algorithm for the Bruhat decomposition and the LDU decomposition for the matrix over the field was obtained in [8], and these algorithms generaized for the matrices over commutative domains was obtained in [9] and [10].

1. Other benefits and application

Control systems.

In 1967 Howard H. Rosenbrock introduced a useful state-space representation and transfer function matrix form for control systems, which is known as the Rosenbrock System Matrix [11]. Since that time, the properties of the matrix of polynomials being intensively studied in the literature of linear control systems.

Groebner basis.

Another important application is the calculation of Gröbner bases. A matrix composed of Buchberger S-polynomials is a strongly sparse matrix. Reduction of the polynomial system is performed when calculating the echelon and diagonal forms of this matrix. The algorithm F4 was the first such matrix algorithm.

Solving PDE's for particle interaction.

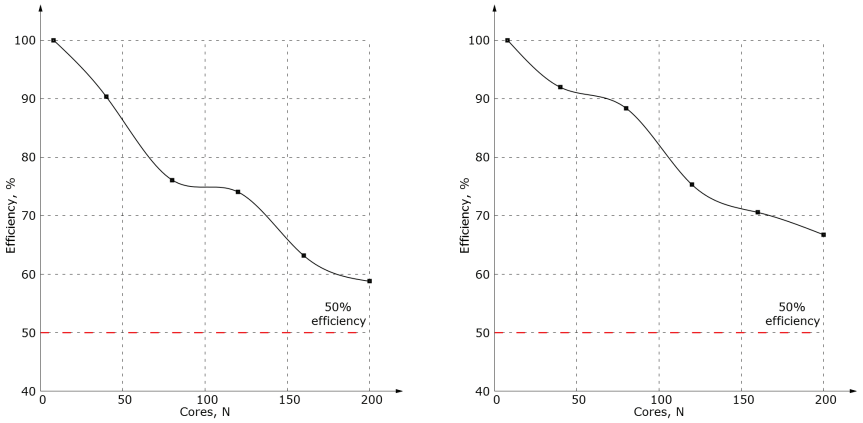
The conservation of the matrix sparseness during the Bruhat decomposition was first investigated in [12]. One of the important class of sparse matrix is called quasiseparable. Any submatrix of quasiseparable matrix entirely below or above the main diagonal has small rank. These quasiseparable matrices arise naturally in solving PDE's for particle interaction with the Fast Multi-pole Method (FMM). The efficiency of application of the block-recursive algorithm of the Bruhat decomposition to the quasiseparable matrices is studied in the article [13].

2. Calculation managing and experiments

The block-recursive matrix algorithms for sparse matrix require a special approachs to managing parallel programs. One approach to the cluster computations management is a scheme with one dispatcher (or one master).

We consider another scheme of cluster menagement. It is a scheme with multidispatching, when each involved computing module has its own dispatch thread and several processing threads[15, 14].

Let us denote by N_i the number of cluster cores and by T_i the computational time in the i-th experiment. For the theoretical best case we'd like to have constant product: $\forall i: \ T_i N_i = \mathbf{const}$. So to demonstrate the efficiency of parallel computational process we have to know the value $E_i = 100\% \ T_i N_i/(T_1 N_1)$. This is the efficiency which was demonstrated in the i-th experiment.



matrix size is 8192 matrix size is 16384

FIGURE 1. The efficiency of the calculation of Schur matrix inversion as a function of the number of cores.

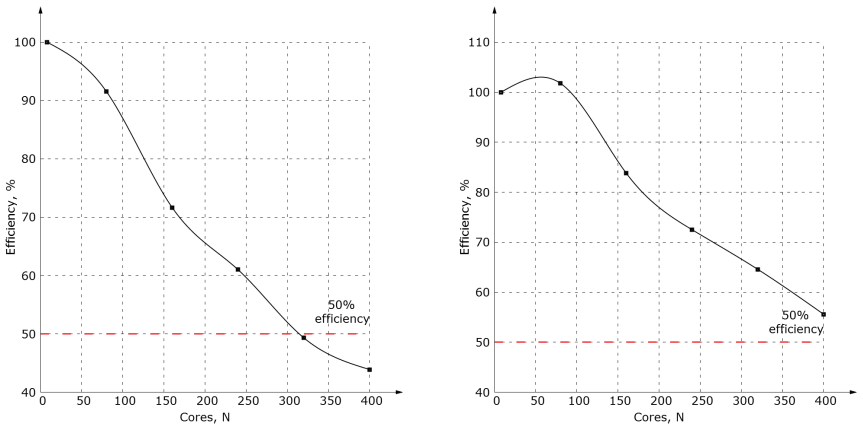
Let $A = \begin{pmatrix} A_0 & A_1 \\ A_2 & A_3 \end{pmatrix}$, $\det(A_0) \neq 0$, $\det(A) \neq 0$, then Schur-Strassen block-recursive matrix inversion is described by formula:

$$A^{-1} = \begin{pmatrix} \mathbf{I} & -A_0^{-1}A_1 \\ 0 & \mathbf{I} \end{pmatrix} \times \begin{pmatrix} \mathbf{I} & 0 \\ 0 & (A_3 - A_2A_0^{-1}A_1)^{-1} \end{pmatrix} \times \begin{pmatrix} \mathbf{I} & 0 \\ -A_2 & \mathbf{I} \end{pmatrix} \times \begin{pmatrix} A_0^{-1} & 0 \\ 0 & \mathbf{I} \end{pmatrix}.$$

For standard matrix multiplication we call it Schur method. And we call it Schur-Strassen method when we used here Strassen's matrix multiplication.

The results of experiments with Schur matrix inversion algorithm are shown in fig. 1. The results for computation of kernel of the matrix operator and adjoint matrix [7] are shown in fig. 2. We use matrices over finite numerical fields in both cases.

The computational experiments was done in Joint Supercomputer Center of the Russian Academy of Sciences (http://www.jscc.ru/scomputers.html) at the cluster mvs10p: Intel Xeon E5-2690, 64G RAM Intel(R) MPI Library for Linux* OS, Version 4.1.0.



matrix size is 8192 matrix size is 12288

FIGURE 2. The efficiency of the calculation of kernel and adjoint matrix as a function of the number of cores.

In experiments we made calculations for identical matrices, using a different number of cluster's nodes.

For the Schur inversin algorithm the minimum number of cores was 8 (1 node), and the maximum number of cores was 200 (25 nodes). For the matrix of size 8192x8192 the number of cores increased 25 times and as a result the calculation time changed 15 times: from 749 sec. to 51 sec. For the matrix of size 16384x16384 number of cores increased 25 times and the calculation time changed 17 times: from 1159 sec. to 69 sec.

For our adjoint and kernell computation algorithm the minimum number of cores was 8 (1 node), and the maximum number of cores was 400 (50 nodes). For the matrix of size 8192x8192 the number of cores in experiment increased 50 times and as a result the calculation time changed 22 times: from 3969 sec. to 181 sec. For the matrix of size 12288x12288 the number of cores increased 50 times and the calculation time changed 28 times: from 15249 sec. to 548 sec.

Static Control of a Parallel Computing Process

One of the most popular approaches in computer algebra is the method of homomorphic images, based on the Chinese remainder theorem. It allows solving the problem simultaneously in many factor rings and using these solutions to obtain a solution of the original problem. For such a process of parallelization, it is sufficient to use static control of calculations on a cluster.

If the elements $m_i, i = 1, .., k$, of the commutative ring R are relatively prime, $m_1..m_k = \mu$, $r_i = x \bmod m_i, i = 1, .., k$, then to find the $x \bmod \mu$ the well-known Newton scheme is usually used.

Below we give Newton scheme for the particular case where the main ring R is a ring of integer numbers. In contrast to the standard scheme of calculations, we only change the order of actions, but we get fewer operations.

Let n_{ij} be inverse element of m_i in Z/m_jZ, i.e. $n_{ij}m_i = 1 \bmod m_j$. Denote:

$$c_1 = r_1, \;\; \bar{c}_1 = c_1 \bmod m_2, \; \nu_1 = n_{1,2} \bmod m_2,$$
$$c_2 = c_1 + m_1(\nu_1(r_2 - \bar{c}_1) \bmod m_2), \; \bar{c}_2 = c_2 \bmod m_3, \; \nu_2 = n_{1,3}n_{2,3} \bmod m_3,$$
$$c_3 = c_2 + m_1m_2(\nu_2(r_3 - \bar{c}_2) \bmod m_3), \bar{c}_3 = c_3 \bmod m_4, \nu_3 = n_{1,4}n_{2,4}n_{3,4} \bmod m_4,$$
$$...$$
$$c_k = c_{k-1} + m_1...m_{k-1}(\nu_{k-1}(r_k - \bar{c}_{k-1}) \bmod m_k), \;\; \bar{c}_k = c_k \bmod m_{k+1},$$
$$\nu_k = n_{1,k+1}n_{2,k+1}..n_{k,k+1} \bmod m_{k+1},$$
$$...$$
$$c_n = c_{n-1} + m_1...m_{n-1}(\nu_{n-1}(r_n - \bar{c}_{n-1}) \bmod m_n).$$

It is easy to verify by induction that the following inequalities hold

$$0 \le c_k < m_1..m_k, \; k = 1, 2, ..n.$$

So the number c_n is not required to be modulo μ, since it is already in the required interval.

Of course, the same scheme can be successfully applied in the ring of polynomials and in all cases when the Chinese remainder theorem holds.

Example:

$$m_i = \{3_1, 5_2, 7_3\}, \; \mu = 105, x = 73,$$
$$r_i = \{1_1, 3_2, 3_3\}, n_{1,2} = 2, n_{1,3} = 5, n_{2,3} = 3, \nu_i = \{2_1, 1_2\},$$
$$c_1 = 1, \; \bar{c}_1 = 1,$$
$$c_2 = 1 + 3(2(3 - 1) \bmod 5) = 13, \; \bar{c}_2 = 6,$$
$$c_3 = 13 + 3 \cdot 5(1(3 - 6) \bmod 7) = 73.$$

References

[1] Dongarra J. With Extrim Scale Computing the Rules Have Changed. In Mathematical Software. ICMS 2016, 5th International Congress, Proceedings (G.-M. Greuel, T. Koch, P. Paule, A. Sommese, eds.), Springer, LNCS, volume 9725, pp. 3-8, 2016.

[2] Strassen V. Gaussian Elimination is not optimal. Numerische Mathematik. V. 13, Issue 4, 354–356 (1969)

[3] Bunch J., Hopkroft J. Triangular factorization and inversion by fast matrix multiplication. Mat. Comp. V. 28, 231-236 (1974)

[4] Malaschonok G. Recursive Method for the Solution of Systems of Linear Equations. Computational Mathematics. A. Sydow Ed, Proceedings of the 15th IMACS World Congress, Vol. I, Berlin, August 1997), Wissenschaft & Technik Verlag, Berlin 1997, 475-480.

[5] Malaschonok G. Effective Matrix Methods in Commutative Domains, Formal Power Series and Algebraic Combinatorics, Springer, Berlin, 2000, 506-517.

[6] Malaschonok G. Matrix computational methods in commutative rings. Tambov, TSU, 2002, 213 p.

[7] Malaschonok G. On computation of kernel of operator acting in a module // Vestnik Tambovskogo universiteta. Ser. Estestvennye i tekhnicheskie nauki [Tambov University Reports. Series: Natural and Technical Sciences], vol. 13, issue 1, 2008. 129-131.

[8] Malaschonok G. Fast Generalized Bruhat Decomposition. Computer Algebra in Scientific Computing, LNCS 6244, Springer, Berlin 2010. 194-202. DOI 10.1007/978-3-642-15274-0_16. arxiv:1702.07242

[9] Malaschonok G. Generalized Bruhat decomposition in commutative domains /in book: Computer Algebra in Scientific Computing. CASC'2013. LNCS 8136, Springer, Heidelberg, 2013, 231-242. DOI 10.1007/978-3-319-02297-0_20. arxiv:1702.07248

[10] Malaschonok G., Scherbinin A. Triangular Decomposition of Matrices in a Domain. Computer Algebra in Scientific Computing. LNCS 9301, Springer, Switzerland, 2015, 290-304. DOI 10.1007/978-3-319-24021-3_22. arxiv:1702.07243

[11] Rosenbrock, H.H. Transformation of linear constant system equations. Proc. I.E.E. V.114, 541–544. (1967).

[12] Kolotilina L., and A. Yemin A. Bruhat decomposition and solution of linear algebraic systems with sparse matrices. Sov.J.Numer.Anal. and Math.Model. v.2, 421–436 (1987).

[13] Pernet C., Storjohann A. Time and space efficient generators for quasiseparable matrices. arXiv:1701.00396 (2 Jan 2017) 29 p.

[14] Ilchenko E.A. An algorithm for the decentralized control of parallel computing process // Vestnik Tambovskogo universiteta. Ser. Estestvennye i tekhnicheskie nauki [Tambov University Reports. Series: Natural and Technical Sciences], 2013, vol. 18, issue 4, 1198-1206.

[15] Ilchenko E.A. About effective methods parallelizing block recursive algorithms // Vestnik Tambovskogo universiteta. Ser. Estestvennye i tekhnicheskie nauki [Tambov University Reports. Series: Natural and Technical Sciences], 2015, vol. 20, issue 5, 1173-1186.

Evgeny Ilchenko
Tambov State University named after G.R.Derzhavin
Tambov, Russia
e-mail: ilchenkoea@gmail.com

Gennadi Malaschonok
Tambov State University named after G.R.Derzhavin
Tambov, Russia
e-mail: malaschonok@gmail.com

Riemann Surfaces and Branch Cuts

David J. Jeffrey

Abstract. The problem of multiply valued functions in computer algebra systems is reviewed. This paper combines the two usual approaches to understanding such functions. The first approach describes the multiple values as *branches* in the complex plane. The second approach describes the multiple values as properties of the Riemann surface of the function. In this paper, a modified implementation in Maple allows us to work with these ideas more efficiently.

1. Introduction

The manner in which computer-algebra systems handle multivalued functions, specifically the elementary inverse functions, has been the subject of extensive discussions over many years. See, for example, [2, 4]. The discussion has centred on the best way to handle possible simplifications, such as

$$\sqrt{z^2} = z \ ? \quad \arcsin(\sin z) = z \ ? \quad \ln(e^z) = z \ ? \tag{1}$$

In the 1980s, errors resulting from the incorrect application of these transformations were common. Since then, systems have improved and now they usually avoid simplification errors, although the price paid is often that no simplification is made when it could be. For example, Maple 18 fails to simplify

$$\sqrt{1-z}\sqrt{1+z} - \sqrt{1-z^2} \ ,$$

even though it is zero for all $z \in \mathbb{C}$, see [2]. Here a new way of looking at such problems is presented.

The discussion of possible treatments has been made difficult by the many different interpretations placed on the same symbols by different groups of mathematicians. Sorting through these interpretations, and assessing which ones are practical for computer algebra systems, has been an extended process. In this paper, we shall not revisit in any detail the many past contributions to the discussion, but summarize them and jump to the point of view taken here.

1.1. A question of values

One question which has been discussed at length concerns the number of values represented by function names. One influential point of view was expressed by Carathéodory, in his highly regarded book [3]. Considering the logarithm function, he addressed the equation

$$\ln z_1 z_2 = \ln z_1 + \ln z_2 \ , \tag{2}$$

for complex z_1, z_2. He commented [3, pp. 259–260]:

> The equation merely states that the sum of one of the (infinitely many) logarithms of z_1 and one of the (infinitely many) logarithms of z_2 can be found among the (infinitely many) logarithms of $z_1 z_2$, and conversely every logarithm of $z_1 z_2$ can be represented as a sum of this kind (with a suitable choice of $\ln z_1$ and $\ln z_2$).

In this statement, Carathéodory first sounds as though he thinks of $\ln z_1$ as a symbol standing for a set of values, but then for the purposes of forming an equation he prefers to select one value from the set. Whatever the exact mental image he had, the one point that is clear is that $\ln z_1$ does not have a unique value, which is in strong contrast to every computer system. Every computer system will accept a specific value for z_1 and return a unique $\ln z_1$.

The reference book edited by Abramowitz & Stegun [1, Chap 4] is another authoritative source, as is its successor [9]. They both define, to take one example, the solution of $\tan t = z$ to be $t = \operatorname{Arctan} z = \arctan z + k\pi$. When listing properties, they both give the equation

$$\operatorname{Arctan}(z_1) + \operatorname{Arctan}(z_2) = \operatorname{Arctan} \frac{z_1 + z_2}{1 - z_1 z_2} \ . \tag{3}$$

For $z_1 = z_2 = \sqrt{3}$, Maple and Mathematica simplify this identity to $\pi/3 = -\pi/3$.

Riemann surfaces give a very pictorial way of seeing multi-valuedness [10, 5], but a question remains whether they can be used computationally [7]. Here we shall continue an approach to Riemann surfaces described by [5] and in references given therein.

2. A new treatment of inverse functions

The basis of the new implementation is notation first introduced in [6]. To the standard function $\ln z$, a subscript is added:

$$\ln_k z = \ln z + 2\pi i k \ .$$

Here the function $\ln z$ denotes the principal value of logarithm, which is the single-valued function with imaginary part $-\pi < \Im \ln z \leq \pi$. This is the function currently implemented in Maple, Mathematica, Matlab and other systems. In contrast, $\ln_k z$ denotes the kth branch of logarithm. With this notation, the statement above of Carathéodory can be restated unambiguously as

$$\exists k, m, n \in \mathbb{Z}, \text{ such that } \ln_k z_1 z_2 = \ln_m z_1 + \ln_n z_2 \ .$$

His “and conversely" statement is actually a stronger statement. He states

$$\forall k \in \mathbb{Z}, \exists m, n \in \mathbb{Z}, \text{ such that } \ln_k z_1 z_2 = \ln_m z_1 + \ln_n z_2 \ .$$

In the light of his converse statement, Carathéodory's first statement could be interpreted as meaning

$$\forall m, n \in \mathbb{Z}, \exists k \in \mathbb{Z}, \text{ such that } \ln_m z_1 + \ln_n z_2 = \ln_k z_1 z_2 \ .$$

This shows the greater conciseness of branch notation.

3. An example function

To save space, we consider only the inverse sine function. The principal branch of the inverse sine function is denoted in Maple by `arcsin`. Using this, we define the branched inverse sine by

$$\operatorname{invsin}_0 z = \arcsin z \ , \tag{4}$$

$$\operatorname{invsin}_k z = (-1)^k \operatorname{invsin}_0 z + k\pi \ . \tag{5}$$

The principal branch now has the equivalent representation $\operatorname{invsin}_0 z = \operatorname{invsin} z = \arcsin z$. It has real part between $-\pi/2$ and $\pi/2$. Notice that the branches are spaced a distance π apart in accordance with the antiperiod[1] of sine, but the repeating unit is of length 2π in accord with the period of sine.

The Maple code for the function is

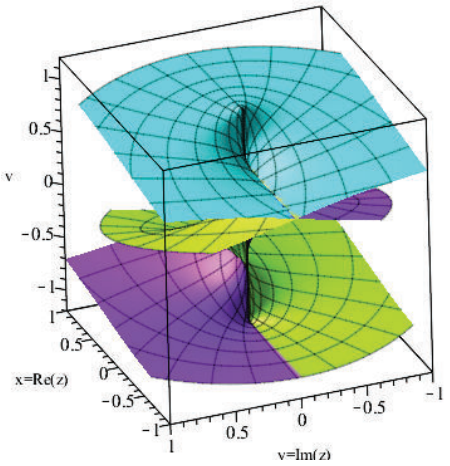
```
invsin := proc (z::algebraic) local branch;
             if nargs <> 1 then
                 error "Expecting 1 argument, got", nargs ;
             elif type(procname, 'indexed') then
                 branch := op(procname);
                 branch*Pi+(-1)^branch*arcsin(z);
             else arcsin(z);
             end if;
          end proc;
```

4. Riemann surfaces

Many multivalued functions are the inverses of single-valued functions. This allows the Riemann surface to be plotted easily. To plot the cube-root surface in Maple, we issue the plot command, which uses the branch notation above for colouring.

```
plot3d([Re((u+I*v)^3),Im((u+I*v)^3), u], u = -1 .. 1, v = -1 .. 1,
colour = (2+k3(u+I*v))*(1/4), view = [-1 .. 1, -1 .. 1, -1 .. 1])
```

[1] An antiperiodic function is one for which $\exists \alpha$ such that $f(z+\alpha) = -f(z)$, and α is then the antiperiod. This is a special case of a quasi-periodic function [8], for which $\exists \alpha, \beta$ such that $f(z+\alpha) = \beta f(z)$.



References

[1] M. Abramowitz, I. J. Stegun. *Handbook of Mathematical Functions*. Dover, 1965.

[2] R. J. Bradford, R. M. Corless, J. H. Davenport, D. J. Jeffrey, S. M. Watt. Reasoning about the elementary functions of complex analysis. *Annals of Mathematics and Artificial Intelligence*, 36:303–318, 2002.

[3] C. Carathéodory. *Theory of functions of a complex variable, 2nd. ed.* Chelsea, New York, 1958.

[4] R. M. Corless and D. J. Jeffrey. The unwinding number. SIGSAM *Bulletin*, 30(2):28–35, June 1996.

[5] R. M. Corless and D. J. Jeffrey. Elementary Riemann surfaces. SIGSAM *Bulletin*, 32(1):11–17, March 1998.

[6] D. J. Jeffrey, D. E. G. Hare, and R. M. Corless. Unwinding the branches of the Lambert W function. *Mathematical Scientist*, 21:1–7, 1996.

[7] D.J. Jeffrey and A.C. Norman. Not seeing the roots for the branches. *SIGSAM Bulletin*, 38(3):57–66, 2004.

[8] Derek F. Lawden. *Elliptic functions and applications*. Springer, 1989.

[9] Daniel W. Lozier, Frank W. J. Olver, and Ronald F. Boisvert. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.

[10] Michael Trott. Visualization of Riemann surfaces of algebraic functions. *Mathematica in Education and Research*, 6:15–36, 1997.

David J. Jeffrey
Dept. of Applied Mathematics
The University of Western Ontario
London, Ontario, Canada
e-mail: djeffrey@uwo.ca

Quantum Behavior in the Framework of Permutation Groups

Vladimir V. Kornyak

Abstract. The trajectory of a quantum system is a sequence of unitary evolutions of vectors in a Hilbert space, interspersed with observations — projections of the vectors in some subspaces, that are specified by measuring devices. Quantum-mechanical description can be made constructive, if we replace the general group of unitary transformations of the Hilbert space by unitary representations of finite groups. It is known that any linear representation of a finite group can be realized as a subrepresentation of some permutation representation. Thus, quantum mechanical problems can be formulated in terms of groups of permutations. Such a constructive approach allows us to clarify the meaning of a number of physical concepts.

1. Quantum Mechanics. Briefly, the formalism of quantum mechanics is reduced to the following. A **pure quantum state** is a ray in a Hilbert space $\mathcal{H}$ over $\mathbb{C}$, i.e. an equivalence class of non-zero vectors in $\mathcal{H}$: $|\psi\rangle \sim a\,|\psi\rangle$, $a \in \mathbb{C}$. The normalization $\langle\psi\,|\,\psi\rangle = 1$ reduces the equivalence: $|\psi\rangle \sim \mathrm{e}^{\mathrm{i}\alpha}\,|\psi\rangle\,,\ \alpha \in \mathbb{R}$. The phase factor $\mathrm{e}^{\mathrm{i}\alpha}$ can be eliminated by replacing the normalized vector $|\psi\rangle$ with the rank one **projector** $\Pi_\psi = |\psi\rangle\langle\psi|$. A weighted mixture of pure quantum states is called a **mixed quantum state**. Any quantum state can be represented by a **density matrix** ρ with characteristic properties: $\rho = \rho^\dagger,\ \rho \geq 0,\ \mathrm{tr}\,\rho = 1$. In particular, for a pure state $|\psi\rangle$ the density matrix is the projector $|\psi\rangle\langle\psi|$. The eigenvalues $p_1, p_2, \ldots$ of a density matrix form a probability distribution in an ensemble of pure states. The Hilbert space of a **composite system** is the *tensor product* of Hilbert spaces for the subsystems: $\mathcal{H}_{XY} = \mathcal{H}_X \otimes \mathcal{H}_Y$. Any mixed state in a Hilbert space $\mathcal{H}$ can be obtained by taking partial trace of a pure state in a "larger" Hilbert space that contains $\mathcal{H}$ as a tensor factor (this is called *"purification"*). **Observation** is detection of the state of a quantum system in one of the mutually orthogonal subspaces that form a partition (defined by an observational setup) of the Hilbert space. The result of quantum observation is random and its statistics is described by a probability measure defined on the subspaces. **Gleason's theorem** states that (for $\dim \mathcal{H} > 2$) any suitable measure has the form $\mu_\rho(S) = \mathrm{tr}\,(\rho\Pi_S)$, where ρ is an arbitrary density matrix, Π_S is the orthogonal projection in the subspace $S \leq \mathcal{H}$. In the case of pure state, $\rho = |\varphi\rangle\langle\varphi|$, and one-dimensional subspace $S = \mathrm{span}\,\{|\psi\rangle\}$, we have the usual **Born rule**: $\mathrm{tr}\,(\rho\Pi_S) \equiv |\langle\varphi\,|\,\psi\rangle|^2$. **Measurement** is a particular case of observation when the orthogonal partition of a Hilbert space is formed by the eigenspaces of an *observable* (an arbitrary Hermitian operator) A. The eigenvalues of A are considered as measured values. The **expectation value** of A in

the state ρ is defined as $\langle A\rangle_\rho = \mathrm{tr}\,(\rho A)$. The **time evolution** of a quantum system between observations is described by a *unitary transformation*: $|\psi_{t'}\rangle = U_{t't}\,|\psi_t\rangle$, or $\rho_{t'} = U_{t't}\rho_t U_{t't}^\dagger$.

2. Constructive Modification of Quantum Formalism. To build constructive models, we need to remove infinities from the formalism. Formally, one can prove that the unitary group $\mathsf{U}(n) \cong \mathrm{Aut}\,(\mathcal{H}_n)$, which has the cardinality of continuum, is empirically equivalent to a **finite group** G. This means that for any particular problem it is always possible to pick a finite group G such that its unitary representation can replace $\mathsf{U}(n)$ without losing the accuracy of describing the empirical data. In essence, it seems more natural to assume that at the fundamental level, it is the finite groups that act, and the continuous unitary groups are only continuum approximations of their unitary representations.

The advantages of finite groups are:

- any finite group is a subgroup of a symmetric group,
- any linear representation of a finite group is unitary and is a subrepresentation of some permutation representation,
- from an empirical point of view, any continuous group can be approximated by a finite one, but not vice versa.

Natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ and **roots of unity** $\mathsf{r} \mid \mathsf{r}^k = 1$ are sufficient to represent all numbers that are significant in the quantum formalism. Let $\mathbb{N}\,[\mathsf{r}_k]$ denote the extension of the semiring $\mathbb{N}$ by a kth primitive root of unity. $\mathbb{N}\,[\mathsf{r}_k]$ is a ring for $k \geq 2$. In particular, this construction gives a way to introduce negative numbers: $\mathbb{Z} = \mathbb{N}\,[\mathsf{r}_2]$. The fraction field of the ring $\mathbb{N}\,[\mathsf{r}_k]$ is the kth **cyclotomic field** $\mathbb{Q}(\mathsf{r}_k)$, which is a dense subfield of $\mathbb{C}$ for $k \geq 3$.

Let G act by permutations on a finite domain Ω, $|\Omega| = \mathsf{N}$. This action induces a **permutation representation** of G on the module $\mathbb{N}^{\mathsf{N}}$ over the semiring $\mathbb{N}$. Any linear representation of G can be realized over a cyclotomic field $\mathbb{Q}\,(\mathsf{r}_k)$, where k is some divisor of the *exponent* of G. Let us extend the module $\mathbb{N}^{\mathsf{N}}$ to the Hilbert space $\mathcal{H}_{\mathsf{N}}$ by extending $\mathbb{N}$ to $\mathbb{Q}\,(\mathsf{r}_k)$. Any **constructive representation** of G can be obtained by projecting the permutation representation of G on the module $\mathbb{N}^{\mathsf{N}}$ (for some sufficiently large N) in an *invariant subspace* of the Hilbert space $\mathcal{H}_{\mathsf{N}}$.

3. Modeling of Quantum Evolution. The time evolution of a quantum system is a sequence of observations with unitary transitions between them. The role of observations is most impressively manifested in the *quantum Zeno effect*. Since the observation of even a pure state leads to a mixed state, it is natural to describe evolution in terms of the density matrix. Suppose that the fundamental ("Planck") time is a sequence of integers: $\mathcal{T} = \{\ldots, 0, 1, 2, \ldots\}$. We define "empirical time" as a sequence of "instants of observations": $t_0, t_1, \ldots, t_{i-1}, t_i, \ldots$. The simplest assumption is that the instants of observations are elements of the fundamental time: $t_i \in \mathcal{T}$. More realistic model of the empirical time would be a distribution around t_i, e.g. the binomial distribution

$$K_\sigma\,(\tau - t_i) = \frac{(2\sigma)!}{4^\sigma (\sigma - t_i + \tau)!\,(\sigma + t_i - \tau)!}, \qquad t_i - \sigma \leq \tau \leq t_i + \sigma. \tag{1}$$

At present, the smallest time uncertainty, i.e. the analog of σ, in physical experiments is about 10^{26} Planck time units. Let ρ_i denote the state of the system *after* the ith observation. According to the standard quantum mechanics, the state *before* the ith observation takes the form $\rho_i' = U\rho_{i-1}U^\dagger$, where $U = \mathrm{e}^{-\mathrm{i}H(t_i - t_{i-1})}$ is a single unitary evolution with a given Hamiltonian H. If ρ_{i-1} is pure, then ρ_i' is also pure. Hamiltonians are usually derived from the principle of least action. Motivated by the fact that any extremal principle involves a selection of dominant elements among many candidates, we propose the following modification: $\rho_i' = \sum_{k=1}^{\mathsf{M}} w_{ik} U_k \rho_{i-1} U_k^\dagger$, where $U_1, \ldots, U_{\mathsf{M}}$ are elements of a unitary representation of a finite group of size M, w_{ik} are (normalized: $\sum_{k=1}^{\mathsf{M}} w_{ik} = 1$) weights of the group elements at the transition $t_{i-1} \to t_i$. A trivial choice of weights reproduces the standard scheme, but in general ρ_i' is a mixed state. The single-step and N-step transition probabilities are $\mathbf{P}_i = \sum_{k=1}^{\mathsf{M}} w_{ik} \operatorname{tr}\left(U_k \rho_{i-1} U_k^\dagger \rho_i\right)$ and $\mathbf{P}_{0\to N} = \prod_{i=1}^{N} \mathbf{P}_i$, respectively. The single-step transition entropy $\Delta\mathbf{S}_i = -\log \mathbf{P}_i$ is a discrete analog of Lagrangian $\mathcal{L}$, and the corresponding entropy of trajectory $\mathbf{S}_{0\to N} = \sum_{i=1}^{N} \Delta\mathbf{S}_i$ is an analog of action $\mathcal{S} = \int \mathcal{L} dt$.

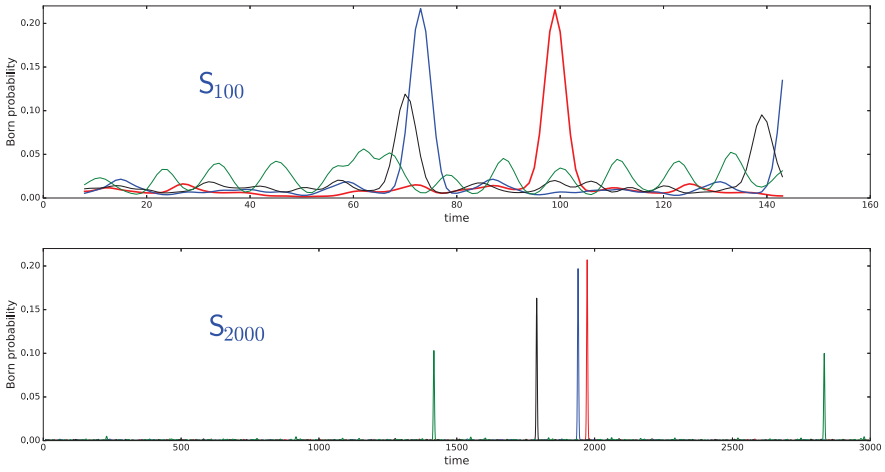
4. The Principle of Least Action as Continuum Approximation. To obtain the continuum limit of the above-described quantum evolution model, we proceed as follows. Replacing the finite group by a Lie group, we introduce the **Lie algebra approximation** in the vicinity of the identity element. For a matrix of unitary representation we have $U \approx \mathbb{1} + \mathrm{i}A$, where A is a Hermitian matrix. We use also the **linear approximation** for the differences of variables by introducing the time derivatives: $X_i - X_{i-1} \approx \dot{X}_i\,(t_i - t_{i-1})$. Discrete sequences are replaced by continuous functions. Note that the case of general mixed states does not allow reasonable continuous approximations: it is natural to assume that the probability of a transition between close density matrices should tend to unity, however $\operatorname{tr}\left(\rho^2\right) = 1$ implies that ρ is a projector. Thus, we shall assume here the case of pure states ψ. All this, together with some standard approximations, leads to the following expression for the Lagrangian $\mathcal{L} = \left\langle \psi \middle| \dot{A}^2 \middle| \psi \right\rangle - \left\langle \psi \middle| \dot{A} \middle| \psi \right\rangle^2 - \mathrm{i}\left(\left\langle \dot{\psi} \middle| \dot{A} \middle| \psi \right\rangle - \left\langle \psi \middle| \dot{A} \middle| \dot{\psi} \right\rangle + 2 \left\langle \psi \middle| \dot{A} \middle| \psi \right\rangle \left\langle \psi \mid \dot{\psi} \right\rangle \right) - \left\langle \psi \mid \dot{\psi} \right\rangle^2$.

5. Searching Dominant Unitary Evolutions in Natural and Standard Representations of Symmetric Group. The usual **natural representation** of $\mathsf{S_N}$ in the Hilbert space $\mathcal{H}_{\mathsf{N}}$ can be obtained from the permutations of coordinates in the module $H = \mathbb{N}^{\mathsf{N}}$ by extending the semiring $\mathbb{N}$ to the field $\mathbb{C}$. The natural representation decomposes into the one-dimensional **trivial** and $(\mathsf{N}-1)$-dimensional **standard** irreducible representations. In fact, any representation of $\mathsf{S_N}$ can be realized over $\mathbb{Q}$ (and over $\mathbb{Z}$). The inner product in the module H is defined as $\langle m \,|\, n \rangle = \sum_{k=1}^{\mathsf{N}} m_k n_k$ for $m, n \in H$. Let $|x\rangle$ denote the normalized natural vector: $|x\rangle = |n\rangle \,/ \sqrt{\langle n \,|\, n \rangle}$. For the natural pure states we have the usual Born rule $\mathbf{P}_{\mathrm{nat}}(x, y) = \langle x \,|\, y \rangle^2$. The space of the standard representation is the $(\mathsf{N}-1)$-dimensional 'standard' subspace in $\mathcal{H}_{\mathsf{N}}$ defined by the condition $x_1 + \cdots + x_{\mathsf{N}} = 0$.

Any vector of the standard subspace can be expressed via the projection of some normalized natural vector. The Born probability in the standard subspace in terms of the normalized natural vectors takes the form

$$\mathbf{P}_{\mathrm{std}}(x,y) = \frac{(\mathsf{N}\langle x\,|\,y\rangle - ab)^2}{(\mathsf{N}-a^2)(\mathsf{N}-b^2)}, \text{ where } a = \sum_{k=1}^{\mathsf{N}} x_k \text{ and } b = \sum_{k=1}^{\mathsf{N}} y_k.$$

Consider the evolution of the initial vector $|x\rangle$, the result of which is measured by the vector $|y\rangle$. The dominant evolutions, i.e. those that provide maximum interaction with the observation device, are of the most interest, since they mainly determine the observed behavior of a quantum system. In the case of the natural and standard representations of the group S_N, all the dominant evolutions for any given pair of vectors $|x\rangle$ and $|y\rangle$ can be found by simple algorithms. The two figures below show the dominant evolutions for four randomly generated pairs of vectors $|x\rangle$ and $|y\rangle$ for the cases $\mathsf{N} = 100$ and $\mathsf{N} = 2000$. The graphs of time evolution of Born's probabilities are smoothed by the binomial kernel (1) with $\sigma = 7$. It is seen that with increasing N, the relative weight of dominant evolutions rapidly increases.



Vladimir V. Kornyak
Laboratory of Information Technologies
Joint Institute for Nuclear Research
Dubna, Russia
e-mail: vkornyak@gmail.com

Applying the Kirchhoff relations in proofs of theorems on graph operations that are not affecting the structure of sandpile group

I. A. Krepkiy

Abstract. The sandpile group of the graph is defined. A series of interesting properties of this group is presented. Theorems on graph operations that do not affect the structure of the sandpile groups is suggested. The Kirchhoff group of graph is defined. The construction of this group is based on relations similar to Kirchhoff laws for electric circuits. The Kirchhoff group of graph is isomorphic to the sandpile group of this graph. New proofs of theorems about graph operations are constructed with usage of this fact.

References

[1] Alexander E. Holroyd, Lionel Levine, Karola Meszaros, Yuval Peres, James Propp, David B. Wilson, *Chip-Firing and Rotor-Routing on Directed Graphs*, available at `http://arxiv.org/abs/0801.3306`.

[2] I. A. Krepkiy, *Sandpile groups and the join of graphs*, Zapiski Nauchn. Semin. POMI 411, 119-124 (2013), available at `http://www.pdmi.ras.ru/znsl/2013/v411/abs119.html`.

[3] I. A. Krepkiy, *Relation between the Sandpile Group of a Graph and its Matroid*, IUS, 3/2015, available at `http://www.slideshare.net/ssuser8d50bc/ss-53219939`.

[4] Baker M., Norine S., *Harmonic morphisms and hyperelliptic graphs*, Int. Math. Res. Notes. 2009. V. 15. P. 2914–2955, available at `https://arxiv.org/abs/0707.1309`.

[5] M. A. Zindinova, I. A. Mednykh, *On the Structure of Picard Group for Moebius Ladder Graph and Prism Graph*, Proceedings of the Fifteenth International Conference on Geometry, Integrability and Quantization, 117–126, Avangard Prima, Sofia, Bulgaria, 2014. doi:10.7546/giq-15-2014-117-126.

I. A. Krepkiy
Mathematics and Mechanics Faculty
St. Petersburg State University
e-mail: `feb418@gmail.com`

MathPartner, как эффективное средство для чтения курса компьютерной алгебры

Геннадий Малашонок

Аннотация. Демонстрируются и обсуждаются новые технологии в образовании, которые появились с приходом облачных систем компьютерной алгебры. В сообщении представлен курс лекций по компьютерной алгебре, который читался в осеннем семестре 2016 года в МПГУ.

Введение. На фоне интенсивного процесса перевода образования на Интернет технологии. интересно понять, что нового могут дать для образования облачные системы компьютерной алгебры, такие как MathPartner [1]-[3].

Облачные системы. Отличие облачных систем от традиционных:

– Облачные системы свободно доступны из любого компьютера или гаджета, который подключен к сети Интернет.

– Они могут обеспечить пользователя развитым сервисом и удобством организации рабочей тетради, так как тетрадь располагается в окне браузера, имеет стандартные функции.

– Исходные тексты математических расчетов на языке Mathpar, которые создаются пользователями, могут свободно сохраняться в Интернете в общедоступных библиотеках. Любой алгоритм из такой библиотеки может быть сразу загружен и исполнен.

— Кроме руководства пользователя по языку Mathpar и страниц помощи с готовыми примерами, имеются меню с подсказками, которые вводят необходимые операторы прямо в текст программы. Рабочее окно имеет два режима: режим исходного текста и режим демонстрации PDF. PDF-файл можно демонстрировать и независимо. Можно сохранить TeX-файл и отдельно компилировать. Анимации, 2D- и 3D- рисунки можно сохраняться на компьютере пользователя.

Модернизация образования. Все это преимущества облачных систем можно уже сегодня использовать в образовании. Но если говорить, например о

школьном образовании, то, конечно необходимо доработать специальный сервис для ученика и учителя, для хранения учебного материала в виде учебников и контрольных работ, автоматической проверки правильности решения задач и выставления оценок.

Настоящее сообщение посвящено новым лекционным технологиям, которые предоставляет Mathpartner.

Лекции, созданные на языке Mathpar, позволяют интенсифицировать образовательный процесс, объединить лекционную и практическую часть.

Язык Mathpar во многом наследовал язык TeX. Он является процедурным языком программирования, операторы которого имеют TeX-подобный синтаксис, а комментарии размещаются внутри двойных кавычек и пишутся на языке TeX. Поэтому лекции можно организовать так, чтобы они учили решать задачи по новому материалу. Теоретическая часть излагается в виде комментариев, а операторы используются для фактического решения задач. Изучив такую лекцию, студент освоит приемы решения задач по новой теме. Он сможет вернуться к такой лекции всегда, когда ему в будущем потребуется решать задачи по этой теме. Такой курс лекция является одновременно и библиотекой алгоритмов для решения задач по данному курсу.

В осеннем семестре 2016 года В МПГУ был прочитан курс лекций по компьютерной алгебре. Центральным разделом курса является алгоритм факторизации полиномов Берлекампа. В результате изучения студент осваивал не только теоретический материал, но и мог самостоятельно выполнить все этапы алгоритма для произвольного входного полинома. В сообщении будет представлен этот курс лекций.

Список литературы

[1] Malaschonok, G.I. Program Comput Soft (2017) 43: 112. doi:10.1134/S0361768817020086 (Малашонок Г.И. Система компьютерной алгебры MathPartner. Программирование, 2017, Т. 43, N. 2.)

[2] Малашонок Г.И. Новое поколение систем символьных вычислений // Вестник ТГУ. Том 21, вып. 6, 2016. с. 2026-2042.

[3] Malashonok G.I. Mathpar Language Guide: Study Guide. Tambov: the Publishing House of TSU named after G.R. Derzhavin, 2013.

Геннадий Малашонок
Тамбовский Государственный университет
Тамбов, Россия
e-mail: malaschonok@gmail.com

Weierstrass Theory of Abelian Integrals and its Realization in Sage

Mikhail D. Malykh and Leonid A. Sevastianov

It is well-known, that abelian integral is an integral of the form

$$\int R(x,y)dx,$$

here R is an arbitrary rational function of the two variables x and y related by the equation

$$f(x,y)=0,$$

where f is an irreducible polynomial from $\mathbb{Q}[x,y]$. Mathematicians of 19th century considered the theory of abelian integrals as the necessary completion of mathematical analysis, many outstanding mathematicians were engaged in development of this theory and its application to differential equations, but after WWI works in this theory have died away. Now in modern CAS there are few packages for work with abelian integrals, probably, `Algcurves` for Maple (M. van Hoeij et all.) is the best of them. This package can calculate, e. g., the genus of a given curve or a basis for the linear space of differentials of the first kind, but can't decompose given integral to integrals of three kinds. Furthermore, for Maple this is a difficult problem even in the case of elliptic curves.

It should be noted that modern algorithms have been created without regard to Weierstrass works [1]. Probably, the matter is that Weierstrass didn't publish his results after 1870 [2]. His lectures on abelian integrals (1875) were published only in 1902 by G. Hettner and J. Knoblauch, so authors of well-known reviews on the theory of abelian integrals (Backer, 1897; Tikhomandritski, 1895) presented Weierstrass's ideas, relying on incomplete students manuscripts. In the 20th century there was a prejudice that Weierstrass theory is not constructive as opposed to Kronecker theory. In our opinion both theories are constructive, but work with different fields, with $\overline{\mathbb{Q}}$ and $\mathbb{Q}$ respectively.

Central notion of Weierstrass theory is a fundamental function for a given curve $f(x,y)=0$. Let H be an element of the field $\mathbb{C}(x,y)$ and equation

$$H(x,y)=t$$

with fixed $t \in \mathbb{P}$ has r roots on the curve, then r is called an order (Grad) of the function H. If (x', y') is a point on this curve then there is such function $H \in \mathbb{C}(x, y)$ that (x', y') is a simple pole of H and the residue at the point is equal 1. Such function with minimal order $r = 1 + p$ is called a fundamental function (Hauptfunktion) and the number p is called a genus (Rang) of curve. In many cases we can write expression for fundamental function explicitly, so for elliptic curve

$$y^2 = a_0 y^3 + a_1 y^2 + a_2 y + a_3$$

fundamental function is equal

$$\frac{1}{2y'} \frac{y + y'}{x - x'},$$

this case has been particularly studied by P.M. Pokrovski [3]. In Weierstrass lectures there is an algorithm for finding the fundamental functions based on power series expansion in local uniformization of a given curve.

Trivial statement of the existence of the fundamental function is the unique existence theorem in Weierstass's lectures, any other objects can be interpreted as derivatives of the fundamental function. By analogy with Green's function, Weierstrass considers not $H(x, y)$, but

$$H(x, y; x', y')dx'.$$

This dual construction is a rational function with respect to (x, y) and a differential with respect to (x', y'). Remarkably it turns out that $H(x, y; x', y')dx'$ with respect to second argument is an integral of the 3rd kind (Art) with the pole at (x, y). If $(a_1, b_1), \ldots, (a_p, b_p)$ are poles of the fundamental function with respect to first argument and power series

$$x = \mathfrak{P}_x(t), \quad y = \mathfrak{P}_y(t)$$

give a local uniformization of a curve in the neighborhood of the point (a_n, b_n), then

$$H(\mathfrak{P}_x(t), \mathfrak{P}_y(t); x', y')dx' = H_n(x', y')dx' \cdot \frac{1}{t} + \cdots - H'_n(x', y')dx' \cdot t + \ldots$$

Coefficients of these expressions give us well-known abelian integrals of the 1st and the 2nd kinds. Green's function is symmetric and by analogy Weierstrass proves that

$$\frac{d}{dx_1} H(x_1, y_1; x_2, y_2) - \frac{d}{dx_2} H(x_2, y_2; x_1, y_1) =$$
$$= \sum_{n=1}^{p} H_n(x_2, y_2) H'_n(x_1, y_1) - H_n(x_1, y_1) H'_n(x_2, y_2),$$

This fundamental equation plays the same role in Weierstrass theory that reduction formulas in the theory of rational integrals. So for any rational function R we can write abelian integral

$$\int R(x, y)dx$$

as sum of algebraic part $R'(x,y)$, log-part

$$\sum_m c_m H(x_m, y_m; x, y)dx$$

with log-singularities in poles of R and the 3rd part

$$\sum_{n=1}^{p} g'_n \int H_n(x,y)dx - g_n \int H'_n(x,y)dx$$

with simple poles in fixed singularities $(a_1, b_1), \ldots, (a_p, b_p)$ of the fundamental function. In lectures by Weierstrass there are explicit formulas for calculation c_m, g_n, g'_n using the power series of local uniformization. This 3nd part is an elementary function iff all g_n and g'_n are equal to zero. So we have:

- decomposition of given abelian integral into integrals of three kinds,
- conditions for integrating given algebraic function in elementary functions, and
- equivalence between Weierstrass definition of genus and commonly used definition as dimension of linear space of homomorphic abelian integrals.

From Weierstrass definition follows that curves with $p = 0$ are biracionally equivalent to the projective line and curves with $p = 1$ are biracionally equivalent to the elliptic curve. Furthermore, we can explicitly write these transformations if we know the fundamental function.

Unfortunately in Weierstrass's lectures there are nocalculation examples, and we want to fill this gap. As it has been noted above the unique instrument of Weierstrass theory is decomposition in Puiseux series over $\mathbb{Q}$. Now in Sage there is a realization of $\overline{\mathbb{Q}}$ (`QQbar`) and polynomial rings over this field written by Carl Witty in 2007. In our talk we want to present some realization of Weierstrass theory in Sage. Russian retelling of Weierstrass's lectures can be found on our site [4].

References

1. *Mark van Hoeij* and *Bernard Deconinck*. Computing Riemann matrices of algebraic curves. // Physica D, 152, 28-46 (2001).
2. *G.I. Sinkevich*. Karl Weierstrass' Bicentenary // Matematika v vysshem obrazovanii. 2015. No 13. C. 143-165.
3. *P.M. Pokrovski*. On rational functions on elliptic curve (in Russian) // matematicheskij sbornik. 1900. Vol. 21.
4. *M.D. Malykh*. Theory of abelian integrals (in Russian) // Materials for the course „Analytical theory of differential equation", MSU, 2002-2012. Url: `https://malykhmd.neocities.org`.

Mikhail D. Malykh
RUDN University
Moscow, Russia
e-mail: malykhmd@yandex.ru

Leonid A. Sevastianov
RUDN University
Moscow, Russia
e-mail: sevast@sci.pfu.edu.ru

Equal Mass Free-Fall Three-Body Problem: Symbolic Dynamics, Numerical Inverstigation

Mylläri Aleksandr, Vassiliev Nikolay and Myullyari Anna

Abstract. We consider equal mass free-fall three-body problem. We construct numerically symbolic sequences using close binary approaches and analyze two components revealed as peaks on the histogram for the Shannon entropy of these sequences.

Introduction

Symbolic dynamics was used to analyze some special cases of the three-body problem: Alexeyev [2, 3, 4, 5] has found an intermittence of motions of different types in the one special case of the three-body problem - Sitnikov problem. Symbolic dynamics was also applied in two other special cases of the three-body problem: the rectilinear problem (Tanikawa & Mikkola [10, 11]); and the isosceles problem (Zare & Chesley [13, 6]). Tanikawa & Mikkola [12] considered the case with non-zero angular momentum.

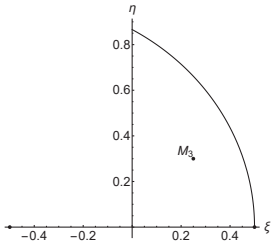


FIGURE 1. Agekian-Anosova region D.

Equal mass free-fall three-body problem is convenient for study since it allows easy visualization of initial configuration: if we place two bodies in the points

$(-0.5; 0)$ and $(0.5; 0)$, then all possible configurations will be covered if we place the third body inside the region D bounded by two straight line segments and arc of the unit circle centered at $(-0.5, 0)$ (Fig. 1) [1]. Here, we analyze components revealed as two peaks on the histogram for the Shannon entropy of the symbolic sequences constructed using close binary approaches that we found earlier [9]. Raspberry Pi cluster was used for numerical integration of trajectories and construction of symbolic sequences, Wolfram Mathematica is used to analyze sequences received. We used symplectic code by Seppo Mikkola (Tuorla Observatory, University of Turku) [8] for numerical simulations.

1. Construction of symbolic sequences

We cannot establish homomorphism between individual trajectories of the three-body system and symbolic sequences in the general case, so we have to integrate equations of motion numerically and construct symbolic sequences during the process.

Final stage of the evolution of typical three-body system is close binary while the third body is ejected from the system. So, all symbolic sequences have predictable final parts. If one will calculate entropy of such "infinite" (long enough in practice) sequence, the result is also obvious. So, we study the evolution of the system during the finite period of time, considering the stage of active interaction between the bodies. Thus, we study complexity of finite sequences. One can say that in our "numerical symbolic dynamics" approach we replace original three-body system by a dynamical system that behaves like our original system during this period of time, and have similar behavior all other time (without disruption).

One can use different methods to construct symbolic sequences (see e.g. [9]). In this study we construct symbolic sequences using binary encounters: we detect minimum distance between two bodies, and corresponding symbol is the number of the distant body. Thus, our symbols are from the alphabet {1, 2, 3}. Some systems disrupt fast, so some sequences are short. Some systems live long (e.g. metastable systems [7]), so corresponding sequences are long. To have a reasonable computing time, we constructed symbolic sequences length 50. Since we are interested in the analysis of active three-body interactions, we consider sub-sequences of each of these sequences, increasing the length step-by-step, calculate entropy for each of these sub-sequences, and find maximum value of these entropies. Maximum value (and moment of time/length of the sub-sequence) correspond to the stage of active interaction between bodies.

Histogram of maximum values of the entropy shows two distinct modes (Fig. 2). Interesting structures can also be seen on the scatterplot of maximum values of the entropy - corresponding length of symbolic sequence in the neighborhood of these modes (Fig. 3). Left mode corresponds to the sequences with only two

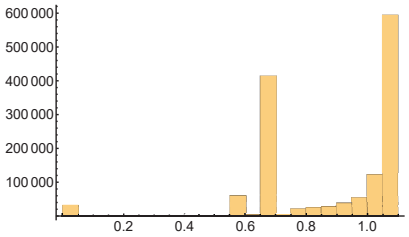


FIGURE 2. Histogram of maximum values of the entropy.

symbols equally represented: Entropy[{1, 2, 1, 2}]=0.693147. Second mode corresponds to the sequences where all three symbols are equally presented: Entropy[{1, 2, 3, 1, 2, 3}]=1.09861.

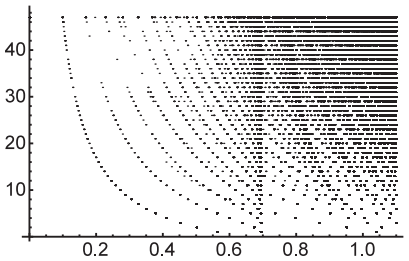


FIGURE 3. Scatterplot of maximum values of the entropy - corresponding length of symbolic sequence.

To reveal the difference between sequences of the type {1,1,1,2,2,2,3,3,3} and {1,2,1,3,2,3,2,1,3} one can use Markov entropy $H = -\sum_i p_i \sum_j q_{ij} \ln q_{ij}$. Here p_i is the frequency of symbol "i" in the sequence, and q_{ij} is the frequency of transitions from "i" to "j". Another option is to form pairs (triples, etc.) of consecutive symbols and calculate corresponding entropies. Histogram of the entropy for sequences constructed from the pairs of symbols for systems from the second invariant component (right mode on Fig. 2) reveals that there is large enough number of systems with good triple interactions (Fig. 4).

Authors acknowledge Dr. Ian V. J. Murray, Dept Physiology and Neuroscience, St. George's University for the collaborative purchase of Wolfram *Mathematica*.

References

[1] Agekian, T.A. and Anosova, J.P. 1967, Astron. Zh., 44, 1261

[2] Alexeyev, V. M. 1968a, Math. sbornik, 76, 72

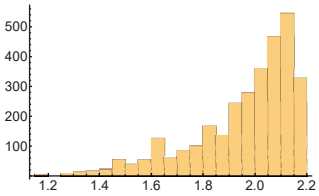


FIGURE 4. Histogram of values of the entropy for pairs of symbols for sequences from the neighborhood of the right mode on Fig. 2.

[3] Alexeyev, V. M. 1968, Math. sbornik, 77, 545

[4] Alexeyev, V. M. 1969, Math. sbornik, 78, 3

[5] Alexeyev, V. M. 1981, Uspekhi math. nauk, 36, 161

[6] Chesley, S. 1999, Celest. Mech. Dyn. Astron., 73, 291

[7] Martynova A.I., Orlov V.V., Rubinov A.V.,2003, MNRAS, 344, 1091

[8] Mikkola, S. and Tanikawa, K. 1999, Celest. Mech. Dyn. Astron., 74, 287-295.

[9] Mylläri, A., Orlov, V., Chernin, A., Martynova, A. and Mylläri, T. 2016, Baltic Astronomy, vol. 25, 254

[10] Tanikawa, K. and Mikkola, S. 2000a, Cel. Mech. Dyn. Astron., 76, 23

[11] Tanikawa, K. and Mikkola, S. 2000b, Chaos, 10, 649

[12] Tanikawa, K. and Mikkola, S. 2007, in the Proceedings of the Workshop held in St. Petersberg, August 25 - 30, 2007, eprint arXiv:0802.2465, 02/2008

[13] Zare, K. and Chesley, S. 1998, Chaos, 8, 475

Mylläri Aleksandr
Dept. of Computers & Technology, SAS
St.George's University
St.George's, Grenada, West Indies
e-mail: amyllari@sgu.edu

Vassiliev Nikolay
V.A. Steklov Institute of Mathematics
of the Russian Academy of Sciences
St. Petersburg, Russia
e-mail: vasiliev@pdmi.ras.ru

Myullyari Anna
Accendo Data LLC
Coral Springs,
Florida, USA
e-mail: anna.myullyari@accendodata.com

Parametrization of Conjugacy Classes Foliation on Lie Group Manifold

Yuri Palii

Consider the action of a classical Lie group on itself by conjugation, i.e.,

$$G \times G \to G, \qquad conj_g(h) = g\,h\,g^{-1}\,. \tag{1}$$

An orbit of this action is the conjugacy class $\mathfrak{C}$ of the group element h. The vector fields tangent to $\mathfrak{C}$ are

$$\zeta_i = \xi_i^r - \xi_i^l\,, \qquad i = 1, \ldots, \dim G\,, \tag{2}$$

where ξ_i^r, ξ_i^l are right and left invariant vector fields on G. The set $\{\zeta_i\}_{i=1}^{\dim G}$ is closed under the Lie bracket operation and the Frobenius theorem asserts that this set defines an integrable (singular) distribution on the Lie group manifold. The corresponding (singular) foliation $\mathcal{F}$ has the conjugacy classes as its leaves.

The maximal tori theorem of E. Cartan says that for each bi-invariant metric Q on G the conjugation orbits intersect every maximal torus T $Q-$orthogonally. Complementary proposition states that every maximal torus of a compact connected Lie group is a section of $\mathcal{F}$, i.e. T meets every conjugacy class $\mathfrak{C}$.

The key property of the foliation $\mathcal{F}$ for us is that its horizontal distribution is also integrable because the tangent space to T is spanned by commuting vector fields forming the Cartan subalgebra. Due to the Frobenius theorem on the language of differential forms, one can find two sets of functions on G:

1. functions which are constant along the orbits, they parametrize the section, i.e. the torus T,
2. fuctions which are constant along the section, they parametrize an orbit, i.e. the conjugacy class $\mathfrak{C}$.

• *Main goal of the work is to find these functions explicitly.*

In the talk a solution of restricted problem will be presented. The space of conjugacy classes (orbits) is isomorphic to the factor of T over the Weyl group W.

So we have a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\subset} & G \\ \downarrow & & \downarrow \\ T/W & \xrightarrow{\cong} & G/conj \end{array}$$

The factor-space $G/conj$ admits a parametrization by polynomial invariants of the conjugation action, e.g., the traces $Tr(g^n),\ g \in G$ where n runs from 1 to the dimension of defining representation. The trace invariants serve as the first set of functions. They define coefficients of the group characteristic equation. The parametrization of the whole torus T needs using roots of the characteristic equation. This step is postponed for the moment.

The second set of functions is found as a solution of PDE-system which express their constancy in terms of Lie derivatives with respect to the vector fields tangent to the section.

The parametrization of conjugacy classes can be related to the same problem for (co)-adjoint orbits in the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. The latest has been intensively discussed many years in the frame of symplectic geometry and integrable systems [1, 2]. Another example of adjoint orbit parametrization is description of the entanglement space of multiqubit systems in mixed state [3] where we need to consider the adjoint action on $\mathfrak{g}$ of a so-called local transformation subgroup in G.

As a possible application area of results we can consider modern string and D-brane models of WZNW-type where a Lie group manifold plays the role of the target space [4]. In these models the vector fields (2) with the 1-form

$$\alpha \sim Tr(\xi(dg\, g^{-1} + g^{-1}\, g)), \qquad \xi \in \mathfrak{g},$$

form the Dirac structure on $TG \oplus T^*G$ and the conjugacy classes play the role of D-branes.

References

[1] M. V. Babich, *Birational Darboux Coordinates on (Co)Adjoint Orbits of $GL(N, \mathbb{C})$*, Funktsional. Anal. i Prilozhen., 50:1 (2016), 20–37.

[2] V. L. Popov, *Rationality of (co)adjoint orbits*, International conference on algebraic geometry, complex analysis and computer algebra (Northern (Arctic) Federal University named after M. V. Lomonosov, Koryazhma, Arkhangelsk region, Russia, August 03–09, 2016), Steklov Math Institute of RAS, Moscow, 2016, 83–85.

[3] V. Gerdt, A. Khvedelidze and Yu. Palii, *Describing orbit space of global unitary actions for mixed qudit states*, Zapiski POMI 421 (2014) 68-80.

[4] A. Alekseev and T. Strobl, *Current algebras and differential geometry*, JHEP 03 (2005) 035.

Parametrization of Conjugacy Classes Foliation

Yuri Palii
Laboratory of Information Technologies,
Joint Institute for Nuclear Research
Dubna, Russian Federation
and
Laboratory of Theoretical Physics,
Institute of Applied Physics,
Chisinau, Republic of Moldova
e-mail: palii@jinr.ru

Factorization Operator Method for Solving BVP Exactly and Finding Eigenvalues and Eigenvectors

I.N. Parasidis and E. Providas

Abstract. We consider a boundary value problem $\mathbf{B}_1 x = f$ where the linear operator $\mathbf{B}_1$ can be decomposed in the form $\mathbf{B}_1 = B_G^2 B_{G_0}^2$ with B_G and B_{G_0} being two linear operators of a special type. If the operator $\mathbf{B}_1$ is correct then the solution can be obtained in closed form. Moreover, the eigenvalues and the eigenvectors of the operator $\mathbf{B}_1$ are computed analytically. A partial integro-differential problem is solved to demonstrate the efficiency of the method.

1. Introduction

The study of many phenomena in science, engineering and economics involve advance mathematical models which in general have a high degree of complexity and they cannot be solved exactly. In these cases powerful numerical methods are usually employed to obtain the solution approximately. Some other problems can be transformed to simpler ones which it is easier to deal with and even to solve them explicitly, see for example [1] and [2] . The present article is a sequel of the work [3] by the same authors as above and discusses the exact solution of a boundary value problem involving an operator factored into two quadratic operators. In particular, we consider the boundary value problem $\mathbf{B}_1 x = f$ where the linear operator $\mathbf{B}_1$ has a decomposition of the form $\mathbf{B}_1 = B_G^2 B_{G_0}^2$ with B_G and B_{G_0} being two linear operators of a special type. We prove that if the operator $\mathbf{B}_1$ is correct then the solution can be obtained in closed form. Moreover, the eigenvalues and the eigenvectors of the operator $\mathbf{B}_1$ can be computed analytically.

We prove a theorem concerning the computation of the determinants of a special class of matrices and two corollaries regarding the evaluation of their eigenvalues and eigenvectors. Next we prove the main theorem for solving boundary value problems involving products of operators. An example problem with an integro-differential operator is chosen to test the efficiency of the method.

2. Special Matrices

Theorem 1. *Let the vectors* $a,\, b \in R^m$ *and the matrix*

$$C = I_m + b^t a.$$

where I_m *stands for the* $m \times m$ *identity matrix and* $b^t = col(b)$*. Then,*

$$|C| = \det(I_m + b^t a) = 1 + ab^t,$$

and for $|C| \neq 0$,

$$C^{-1} = \frac{1}{|C|}(|C| I_m - b^t a) = I_m - \frac{1}{|C|} b^t a.$$

□

From Theorem 1 some other results can be derived which are contained in the next two corollaries and are used in the present paper.

Corollary 1. *Let* $a, b \in R^m$*, the matrix* $C = I_m + b^t a$*, and* $|C| = \det C$*. Then the next statements are true:*
(i) The number $|C|$ *is an eigenvalue of the matrix* C *and* b^t *its corresponding eigenvector, namely*

$$Cb^t = (I_m + b^t a)b^t = |C| b^t.$$

(ii) If $|C| \neq 0$*, then the number* $\frac{1}{|C|}$ *is an eigenvalue of the matrix* C^{-1} *and* b^t *is its corresponding eigenvector, explicitly*

$$C^{-1} b^t = \frac{1}{|C|}\big(|C| I_m - b^t a\big) b^t = \frac{1}{|C|} b^t.$$

(iii) The number $|C|$ *is an eigenvalue of the matrix* C^t *and* a^t *is its corresponding eigenvector, that is to say*

$$aC = a|C| \quad or \quad C^t a^t = |C| a^t.$$

(iv) If $|C| \neq 0$*, then the number* $\frac{1}{|C|}$ *is an eigenvalue of the matrix* $(C^t)^{-1}$ *and* a^t *is its corresponding eigenvector, specifically*

$$aC^{-1} = \frac{1}{|C|} a \quad or \quad (C^t)^{-1} a^t = \frac{1}{|C|} a^t.$$

(v)If $|C| \neq 0$ *then,*

$$1 - aC^{-1} b^t = \frac{1}{|C|}.$$

□

Corollary 2. *Let the vectors* $a,\, b,\, c \in R^m$, $k \in \mathbb{C}$ *and the matrix*

$$C_1 = \begin{pmatrix} I_m + kb^t a & kb^t c \\ b^t a & I_m + b^t c \end{pmatrix} = I_{2m} + \begin{pmatrix} kb^t \\ b^t \end{pmatrix} \begin{pmatrix} a & c \end{pmatrix}.$$

Then, $\det C_1 = 1 + (ka + c)b^t$ *and for* $\det C_1 = |C_1| \neq 0$,

$$\left(\begin{array}{cc} a & c \end{array} \right) C_1^{-1} = \frac{1}{|C_1|} \left(\begin{array}{cc} a & c \end{array} \right) \quad and \quad 1 - \left(\begin{array}{cc} a & c \end{array} \right) C_1^{-1} \left(\begin{array}{c} kb^t \\ b^t \end{array} \right) = \frac{1}{|C_1|},$$

$$C^{-1} = \frac{1}{|C_1|} \left(\begin{array}{cc} |C_1| I_m - kb^t a & -kb^t c \\ -b^t a & |C_1| I_m - b^t c \end{array} \right) = I_{2m} - \frac{1}{|C_1|} \left(\begin{array}{c} kb^t \\ b^t \end{array} \right) \left(\begin{array}{cc} a & c \end{array} \right).$$

□

3. Factorization Operator Method

We cite now the main theorem of the current work.

Theorem 2. *Let the operator* $\mathbf{B}_1 : H \to H$ *be defined by*

$$\begin{aligned} \mathbf{B}_1 x &= \widehat{A}^2 \widehat{A}_0^2 x - V \langle \widehat{A}_0 x, \Phi^t \rangle_{H^m} - Y \langle \widehat{A}_0^2 x, \Phi^t \rangle_{H^m} - S \langle \widehat{A} \widehat{A}_0^2 x, F^t \rangle_{H^m} \\ &\quad - G \langle \widehat{A}^2 \widehat{A}_0^2 x, F^t \rangle_{H^m} = f, \quad D(\mathbf{B}_1) = D(\widehat{A}^2 \widehat{A}_0^2), \end{aligned} \tag{1}$$

where $\widehat{A}_0$, $\widehat{A} : H \to H$ *are linear correct operators and the vectors* $F, \Phi \in H^m$. *We also suppose that* x_0 *is an eigenvector of both operators* $\widehat{A}_0$ *and* $\widehat{A}$, *the numbers* α_0, α *are the corresponding to* x_0 *nonzero eigenvalues of the operators* $\widehat{A}_0$ *and* $\widehat{A}$, *respectively. Finally, let* $Y = (y_1 x_0, ..., y_m x_0) = y x_0$, $G = (g_1 x_0, ..., g_m x_0) = g x_0$ *and* $S, V \in H^m$, *where* $y_i, g_i \in \mathbb{C}$, $i = 1, ..., m$. *Then the following statements hold:*
(i) If

$$S = \alpha D g x_0, \quad V = \alpha_0 D_0 y x_0, \tag{2}$$

where $D_0 = 1 - \dfrac{1}{\alpha^2 D^2} y \langle x_0, \Phi^t \rangle_H$, *and* $D = 1 - g \langle x_0, F^t \rangle_H \neq 0$, *then,* $\mathbf{B}_1$ *has the unique decomposition*

$$\mathbf{B}_1 = B_G^2 B_{G_0}^2, \tag{3}$$

where

$$B_{G_0} x = \widehat{A}_0 x - G_0 \langle \widehat{A}_0 x, \Phi^t \rangle_{H^m} = f, \quad D(B_{G_0}) = D(\widehat{A}_0), \tag{4}$$

$$B_G x = \widehat{A} x - G \langle \widehat{A} x, F^t \rangle_{H^m} = f, \quad D(B_G) = D(\widehat{A}), \tag{5}$$

with $G_0 = \dfrac{1}{\alpha^2 D^2} y x_0$.
(ii) If G_0, S, V *satisfy (2) then* x_0 *is also the eigenvector of the operators* B_G, B_{G_0} *and* $\mathbf{B}_1 = B_G^2 B_{G_0}^2$, *while the numbers* $\alpha D, \alpha_0 D_0$ *and* $\alpha^2 \alpha_0^2 D^2 D_0^2$ *are the corresponding eigenvalues of* B_G, B_{G_0} *and* $\mathbf{B}_1 = B_G^2 B_{G_0}^2$, *respectively.*
(iii) If (2) is true and in addition $\widehat{A}_0, \widehat{A}$ *are densely defined then* $\mathbf{B}_1$ *is correct if and only if the number* $D_0 \neq 0$.
(iv) If (2) is valid and $\mathbf{B}_1 = B_G^2 B_{G_0}^2$ *is correct then the unique solution of the*

problem (1) is given by

$$\begin{aligned} x = \mathbf{B}_1^{-1} f &= \widehat{A}_0^{-2}\widehat{A}^{-2} f \\ &+\frac{x_0}{\alpha_0^2\alpha^2 D_0^2 D^2}\Big[\alpha D g\langle \widehat{A}^{-1} f, F^t\rangle_{H^m} + g\langle f, F^t\rangle_{H^m} \\ &+y\langle \widehat{A}^{-2} f, \Phi^t\rangle_{H^m} + \alpha_0 D_0 y\langle \widehat{A}_0^{-1}\widehat{A}^{-2} f, \Phi^t\rangle_{H^m}\Big]. \quad \square \end{aligned} \tag{6}$$

4. Applications

In what follows $H^i(\Omega)$ denotes the Sobolev space of all complex functions of $H = L_2(\Omega)$ which have generalized derivatives up to the i order that are Lebesque integrable, $i = 1, 2, 3, 4$, $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \le x, y \le 1\}$.

Example 3. Consider the operator $\mathbf{B}_1 : L_2(\Omega) \to L_2(\Omega)$ defined by

$$\begin{aligned} \mathbf{B}_1 u &= u_{xxyy} + V\int_0^1\int_0^1 (2x-1)u_x dydx \\ &+i\pi e^{-i\pi(x+y)}\int_0^1\int_0^1 (2x-1)u_{xx}dxdy + S\int_0^1\int_0^1 (y^2-y)u_{xxy}dxdy \\ &-2\pi^2 e^{-i\pi(x+y)}\int_0^1\int_0^1 (2y-1)u_{xxy}dxdy = f, \\ D(\mathbf{B}_1) &= \{u \in H^4(\Omega) :\ u(0,y) + u(1,y) = 0,\ u_x(0,y) + u_x(1,y) = 0, \\ &\qquad u_{xx}(x,0) + u_{xx}(x,1) = 0,\ u_{xxy}(x,0) + u_{xxy}(x,1) = 0\}, \end{aligned} \tag{7}$$

where V, S are unknown functions of $L_2(\Omega)$. By Theorem 2 we have:
(i) If

$$S = 2\pi i(\pi^2 - 16)e^{-i\pi(x+y)}, \quad V = \frac{i\pi^2(\pi^2 - 32\pi + 264)}{(\pi^2-16)^2}e^{-i\pi(x+y)}, \tag{8}$$

then $\mathbf{B}_1$ is correct and has the unique decomposition $\mathbf{B}_1 = B_G^2 B_{G_0}^2$, where B_{G_0}, B_G are defined by (4)-(5), respectively and $G_0 = \frac{i\pi^3}{\pi^2+16}e^{-i\pi(x+y)}$,
(ii) If S, V satisfy (8) then the number $\beta = \frac{(\pi^4+32\pi^2+264)^2}{(\pi^2+16)^2}$ is an eigenvalue of $\mathbf{B}_1$ and $x_0 = e^{-i\pi(x+y)}$ its corresponding eigenvector,
(iii) If S, V satisfy (8) then the unique solution of (7) is given by

$$\begin{aligned} x &= \mathbf{B}_1^{-1} f = \widehat{A}_0^{-2}\widehat{A}^{-2} f + \frac{x_0}{\beta}\Big[\frac{i\pi(\pi^2+16)}{6}\int_0^1\int_0^1 (4y^3 - 6y^2 + 1)f(x,y)dxdy \\ &-2\pi\int_0^1\int_0^1 (y^2-y)f(x,y)dxdy + \frac{i\pi}{2}\int_0^1\int_0^1 (2x-1)(y-y^2)f(x,y)dxdy \\ &-\frac{\pi^2(\pi^4+32\pi^2+264)}{2(\pi^2+16)^2}\int_0^1\int_0^1 (x^2-x)(y-y^2)f(x,y)dxdy\Big]. \end{aligned}$$

where

$$\begin{aligned}\widehat{A}_0^{-2}\widehat{A}^{-2}f &= \int_0^x (x-x_1)dx_1 \int_0^y (y-y_1)f(x_1,y_1)dy_1 \\ &\quad -\frac{1}{4}\int_0^x (x-x_1)dx_1 \int_0^1 (2y-2y_1+1)f(x_1,y_1)dy_1 \\ &\quad -\frac{1}{4}\int_0^1 (2x-2x_1+1)dx_1 \int_0^y (y-y_1)f(x_1,y_1)dy_1 \\ &\quad +\frac{1}{16}\int_0^1 (2x-2x_1+1)dx_1 \int_0^1 (2y-2y_1+1)f(x_1,y_1)dy_1.\end{aligned}$$

References

[1] L.M. Barkovsky and A.N. Furs, *Factorization of integro-differential equations of the acoustics of dispersive viscoelastic anisotropic media and the tensor integral operators of wake packet velosities*. Acoustical Physics. 48 (2002), no. 2., 128-132.

[2] E.A. Khachatryan, *Solvability of an Integro-Differential Equation with Partial Derivatives on the Half-Axis*, Journal of Contemporary Mathematical Analysis. 43 (2008), no. 5., 60-69.

[3] I.N. Parasidis, E. Providas and P.C. Tsekrekos, *Factorization of linear operators and some eigenvalue problems of special operators*. Vestn. Bashkir. Univ. Russia. 17 (2012), no. 2., 830-839.

I.N. Parasidis
Dept. of Electrical Engineering
TEI of Thessaly
Larissa, Greece
e-mail: `paras@teilar.gr`

E. Providas
Dept. of Mechanical Engineering
TEI of Thessaly
Larissa, Greece
e-mail: `name2@xyz.com`

Applications of Steiner symmetrization to some extremal problems in geometric function theory

Ronen Peretz

We investigate properties of the Steiner symmetrization in the complex plane. We use two recursive dynamic processes in order to derive some sharp inequalities on analytic functions in the unit disk. We answer a question that was asked by Albert Baernstein II, regarding the coefficients of circular symmetrization. We mostly deal with the Steiner symmetrization G of an analytic function f in the unit disk U. We pose few problems we can not solve. An intriguing one is that of the inequality

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |G(re^{i\theta})|^p d\theta,\ 0 < p < \infty$$

which is true for $p = 2$ (we prove) but can not be true for too large p. What is the largest such exponent or its supremum?

Ronen Peretz
Department of Mathematics
Ben Gurion University of the Negev
Beer-Sheva , 84105
Israel
e-mail: ronenp@math.bgu.ac.il

Zero divisors in group rings and combinatorial corollaries

Fedor Petrov

Let G be a finite group, K a field and $X_1, \dots, X_k$ be subspaces of the group algebra $K[G]$ such that $X_1 \dots X_k = 0$. If the codimensions of X_i are small, it implies many strong combinatorial properties of G, like the following:

any product set $A_1 \dots A_{k-1}$ may be covered by a set C of size $\operatorname{codim} X_k$ and $k-1$ smaller product subsets $A_1 \dots A_{i-1} B_i A_{i+1} \dots A_{k-1}$ for certain $B_i \subset A_i$, $|B_i| \leqslant \operatorname{codim} X_i$. This generalizes the previous results by Ellenberg and Kleinberg, Sawin, Speyer, which all originated in a breakthrough paper by Croot, Lev and Pach.

Fedor Petrov
Sankt-Petersburg Department
Steklov Institute of Mathematics
191023, Fontanka 27,
Sankt-Petersburg, Russia
e-mail: fedyapetrov@gmail.com

Reuse of educational tasks that support e-learning automation

Ilya Posov

Educational tasks are used by teachers and students to monitor and evaluate their learning. The features of tasks may vary a lot, for example, tasks may contain some or all of the following: a text with a statement, an answer, automatically checked answers, feedback for student responses, hints, generation of statements, interactive interface, etc. Most of these features need specialized learning software, and usually tasks prepared for one learning system are not supported by other systems. If a teacher has a large tasks repository in one learning system, that does not fully fit his or her needs, it is usually impossible to reuse this repository in another learning system. This may lead to usage of several learning systems with different tasks sets.

Ideally, a teacher has one tasks repository and a set of software tools used in different educational situations. Each tool supports a subset of tasks, and the more intersections these subsets have, the more useful actions a teacher may perform with an average task.

This may be achieved if we consider a task as an object in terms of the object oriented paradigm. A task exposes several interfaces that it implements. For example, a task may have actions to get its statement for displaying to a student, to grade a student's response, etc. Many e-learning automation routines become a responsibility of a task, and not of a learning system. For example, the generation of a statement is done by a task when a learning system asks for its statement; an answer grading is done by a task by either comparing the answer with a hard-coded correct one, or by running a computer algebra system with checking computations. The task is represented as a structured data in a file or a byte stream, with the information about exposed interfaces, so that a learning system may discover task's abilities and decide, whether it supports a task or not. The implementation of interfaces may be done inside of the problem, but tasks usually exists as sets of similar tasks, so the implementation may move out of the task to not repeat itself.

The tasks and learning systems, that use them, are not the only objects in the proposed architecture. One also need adapters that implement interfaces by means

of other interfaces. For example, tasks from a bebras contest in informatics have many information slots such as a statement, a question, a list of possible answers, an explanation, definitions of concepts, connection with informatics, etc. These are exposed as an interface of a task for a bebras contest, and only the bebras contest system may understand all this information. But if we have an adapter that converts this interface to the interface that allows to get tasks statement, we may use bebras problems in almost any learning system, because they will be able to show their statement to students, that is sometimes enough.

The proposed architecture is similar to audio and video processing software: there are many processing tools, there are many audio and video files of different formats, and there are codecs that make these formats understandable for processing tools.

The implementation of a proposed architecture needs to address many problems, that are well-known because the idea of stand-alone object, i.e. objects that exist outside of the software, is not new. Some of them are about a setup of an environment, security, performance: a learning system may not expect that getting a statement sometimes takes too much time because this operation needs an access to a computer algebra system. The report presents an overview of the architecture with the discussion of how to solve the problems and make the system usable for at least the types of tasks that the author works with. They include tasks for different types of online and offline contests, research laboratories, tasks for students for different school and university courses (mathematics and programming in general purpose programming languages and mathematical software), many of task types are generable and suppose automatic grading.

Ilya Posov
Faculty of Arts
St. Petersburg State University,
Saint-Petersburg, Russia,

Faculty of Computer Science and Technology
St. Petersburg Electrotechnical University "LETI"
Saint-Petersburg, Russia
e-mail: i.posov@spbu.ru
e-mail: iposov@gmail.com

Computer algebra system as a pedagogical task

Sergei Pozdniakov

Teaching of algorithmic mathematics within a course of discrete mathematics include acquaintance of students with algorithms on long numbers and polynomials and their application for solution of various applied tasks, for example in cryptography.

Within a course of mathematics there is no opportunity to discuss and especially to realize all such algorithms.

Within three years The Mathematics Department of SPbETU "LETI" makes the experiment on creation of computer mathematics systems consisting from more than 40 interacting modules by students teams. Every year about 200 students who are divided into teams of 6-20 members participate in the project.

The project has two main goals:

1. to connect knowledges of programming with knowledges on algorithmic mathematics
2. to learn to work in teams, to optimise distributing of works among themselves and to make decisions on architecture of system.

As a rule, the team consists of one academic group, but it is allowed both to unite with other groups and to separate into small groups. Management of work of each team is provided by two persons: architect of system of computer algebra and responsible for quality management.

The architect of each team receives the description of structure of system: names and functions of all modules and description of links between them. After that he/she chooses structure of data, programming language, system of versions support (for example, GitHub), specifies interfaces for interaction of modules and defines a type of user interface, and also the requirement to readability of separate modules and technology of testing.

Responsible for quality management organises contact of students with the architect so that each student of group benefited from the project; after project terminates responsible for quality management creates an anonymous questionnaire to which all members of team respond and which gives the answer to whether they succeeded in distribution of works between students - whether well they understood sense of the common work. Also students which are responsible for quality

management from all the teams, in unison define metrics for estimation of software quality.

3-4 weeks are allocated for all works. The most inspiring stage of the project - presentation them to other teams and discussion. Complete projects are placing in common access for preliminary acquaintance with them members of other teams. The projects must be presented by incidentally chosen students from each team (in total about 15 teams) which are given 10 minutes for demonstration and 5 minutes for answers of architects and other students from all the teams. Students very much worry about quality of the projects and in spite of the fact that they are not considered formally in progress, try to finalise them and optimize.

Results of questioning of students show that in average each student works about 1–3 hours and that more time is needed for linking of modules and correction of mistakes, than on their coding. Most of students (about 80%) consider this work as the useful and interesting.

In other questionaries next year students often mark experience on creation of systems of computer algebra as the example which they remember and consider as interesting and useful form of studying.

Sergei Pozdniakov
Faculty of Computer Science and Technology
St. Petersburg Electrotechnical University "LETI"
Saint-Petersburg, Russia
e-mail: pozdnkov@gmail.com

On zeta functions of categories

N. V. Proskurin

We consider possible ways to attach some kind of zeta functions to categories. At present, three different definitions were suggested.

(1) Kazunori Noguchi (Documenta Mathematica 18, pp. 1243–1274, 2013) defines zeta functions of finite categories. These zeta functions are somewhat similar to that of varieties over finite fields.

(2) Nobushige Kurokawa (Proc. Japan Acad., Ser. A, **60**, 1984, No. 9, pp. 335–338; **72**, 1996, No. 10, pp. 221–222) defines zeta functions of categories by means of the Euler products those local factors are similar to that of the Riemann zeta function.

(3) One more definition is given recently by the author (Zap. nauchn. seminarov POMI, **449**, pp. 230-234, 2016). From our viewpoint, we can define zeta functions of categories by means of Dirichlet series and we should not state form of local factors and existence of the Euler products just in the definition.

We intend to look in details on these definitions, to state basic properties of the zeta functions, and to give some samples.

N. V. Proskurin
Sankt-Petersburg Department
Steklov Institute of Mathematics
191023, Fontanka 27,
Sankt-Petersburg, Russia
e-mail: np@pdmi.ras.ru

An Approach to the Set Partition Problem

Alexandr V. Seliverstov

Abstract. The article focuses on methods to confirm the smoothness of some cubic hypersurfaces that are closely related to the set partition problem.

Let us recall the set partition problem [1]. Given a multiset of positive integers $\{\alpha_0, \ldots, \alpha_n\}$. Can it be partitioned into two subsets with equal sums of elements? Points with coordinates ± 1 are called $(-1, 1)$-points. The problem is to recognize whether a $(-1, 1)$-point belongs to the hyperplane given by the linear equation $\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n = 0$. It is hard to find a $(-1, 1)$-point belonging to the hyperplane in high dimensions [2]. On the other hand, one can find $(-1, 1)$-points belonging to the hyperplane given by a linear function with integer coefficients near zero, using dynamic programming [1].

Let us consider an affine hypersurface $\mathcal{F}$ that is the vanishing locus of a square-free polynomial f. A straight line passing through the selected point $U \in \mathcal{F}$ is the set of points with coordinates $((x_1 - u_1)t + u_1, \ldots, (x_n - u_n)t + u_n)$, where $(u_1, \ldots, u_n)$ are coordinates at U, and t is a parameter. Let us denote by $r(t)$ a univariate polynomial that is the restriction of the polynomial f to the line, and by $B[f, U]$ the discriminant of $r(t)/t$. Since $r(0) = 0$, $r(t)/t$ is a polynomial of degree at most $d-1$, where $d = \deg f$. If $\deg r(t) < d-1$, then we use the formula for degree $d-1$ by means of substitution the zero as the leading coefficient. If the point U is smooth, then $B[f, U](x_1, \ldots, x_n)$ defines a cone.

Let us denote by $\mathbb{K}$ a finite extension of the field of rational numbers $\mathbb{Q}$. Any smooth cubic curve is not unirational. In accordance with [3], for each cubic surface as well as high dimensional hypersurface $\mathcal{X}$ defined over $\mathbb{K}$, if $\mathcal{X}$ is irreducible, $\mathcal{X}$ is not a cone, and $\mathcal{X}$ contains a $\mathbb{K}$-point, then $\mathcal{X}$ is unirational over $\mathbb{K}$. That is, we have not only a lot of $\mathbb{K}$-points but also a rational map from the set of $\mathbb{Q}$-points of the affine space to the set of $\mathbb{K}$-points of $\mathcal{X}$. The explicit parameterizations of the Clebsch diagonal surface as well as the Fermat cubic surface are exemplified in [4]. Both surfaces are rational over $\mathbb{Q}$.

Let us denote $f = \alpha_0 + \alpha_1 x_1^3 + \cdots + \alpha_n x_n^3$ and $h = \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n$, where all coefficients $\alpha_0, \ldots, \alpha_n$ are nonzero. $\mathcal{F}$ denotes the affine cubic hypersurface given

by the equation $f = 0$ as well as $\mathcal{H}$ denotes the hyperplane given by $h = 0$. The following theorem improves the result from [5] in case of cubic hypersurfaces.

Theorem 1. *Given a multiset of positive integers $\{\alpha_0, \dots, \alpha_n\}$. There exists a one-to-one correspondence between singular points of the hyperplane section $\mathcal{F} \cap \mathcal{H}$ and $(-1, 1)$-points belonging to the hyperplane $\mathcal{H}$.*

Proof. If both polynomials f and h vanish simultaneously at a $(-1, 1)$-point, then the hyperplane $\mathcal{H}$ is tangent to the hypersurface $\mathcal{F}$ at this point. Thus, the hyperplane section is singular. Contrariwise, at a singular point of the section, the hyperplane $\mathcal{H}$ coincides with the tangent hyperplane to the hypersurface $\mathcal{F}$. Since all the coefficients α_k are nonzero, both gradients ∇f and ∇h can be collinear only at the points whose coordinates satisfy the system of the equations $x_k^2 = x_j^2$ for all indices k and j. All the points are $(-1, 1)$-points. □

The polynomial $B[f, U]$ is equal to the discriminant of a univariate polynomial $at^2 + bt + c$. That is, $B[f, U] = b^2 - 4ac$, where the coefficients are sums of univariate polynomials $a = a_1(x_1) + \dots + a_n(x_n)$, $b = b_1(x_1) + \dots + b_n(x_n)$, and $c = c_0 + c_1x_1 + \dots + c_nx_n$. Each monomial from $B[f, U](x_1, \dots, x_n)$ is dependent on at most two variables.

Let us consider the factor ring $\mathbb{K}[x_1, \dots, x_n]/\langle x_1^2 - 1, \dots, x_n^2 - 1\rangle$. It is referred to as the set of multilinear polynomials. In this way, we have a surjective map φ from the set of all polynomials onto the set of multilinear polynomials.

Let us denote by $M[f, U](x_1, \dots, x_{n-1})$ a multilinear polynomial that is an image of the restriction to the hyperplane $\mathcal{H}$ of the multilinear polynomial $\varphi(B[f, U])$. The restriction to the hyperplane $\mathcal{H}$ means that we substitute $x_n = -(\alpha_0 + \alpha_1x_1 + \dots + \alpha_{n-1}x_{n-1})/\alpha_n$. Let us denote by $\mathcal{L}$ a linear space spanned by all multilinear polynomials $M[f, U](x_1, \dots, x_{n-1})$, where $U \in \mathcal{F} \cap \mathcal{H}$.

A polynomial vanishes at a $(-1, 1)$-point if and only if its multilinear image vanishes at this point. Thus, if the hyperplane $\mathcal{H}$ contains a $(-1, 1)$-point, then all multilinear polynomials from $\mathcal{L}$ vanish at the point. Contrariwise, if a nonzero constant belongs to the linear space $\mathcal{L}$, then $\mathcal{H}$ does not contain any $(-1, 1)$-point. In the case, $\mathcal{F} \cap \mathcal{H}$ is smooth.

In case $n = 2$, let us consider values $\alpha_0 = 1$, $\alpha_1 = 3$, and $\alpha_2 = 2$. The intersection $\mathcal{F} \cap \mathcal{H}$ consist of two points $U(-1, 1)$ and $V(\frac{1}{5}, -\frac{4}{5})$. The multilinear polynomial $\varphi(B[f, U]) = -72x_2x_1 - 48x_2 - 144x_1 - 168$. The substitution $x_2 = -\frac{3x_1+1}{2}$ yields a univariate polynomial $108x_1^2 - 36x_1 - 144$. Its multilinear image $M[f, U] = -36x_1 - 36$. At the second point V the multilinear polynomial

$$M[f, V] = \frac{26172}{3125}x_1 + \frac{428292}{15625}.$$

Two polynomials $M[f, U]$ and $M[f, V]$ together span the whole linear space of univariate linear polynomials. The same holds for almost all values α_1 and α_2 because $\dim \mathcal{L}$ is a lower semi-continuous function.

Contrariwise, if $n = 2$ and $\alpha_0 = \alpha_1 = \alpha_2 = 1$, then $\dim \mathcal{L} = 1$. The intersection $\mathcal{F} \cap \mathcal{H}$ consist of two points $U(0, -1)$ and $V(-1, 0)$. The third point does

not belong to the affine plane. So, $B[f,U] = -12x_1x_2 - 24x_2 - 12x_1 - 24$; the multilinear polynomial $M[f,U] = 24x_1 + 12$. On the other hand, at the point V the polynomial $B[f,V] = -12x_1x_2 - 12x_2 - 24x_1 - 24$; the multilinear polynomial $M[f,V]$ vanishes identically. Thus, the linear space $\mathcal{L}$ is a proper subspace in the two-dimensional space of univariate linear polynomials.

If $n = 4$ and $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$, then $\dim \mathcal{L} = 1$. The space $\mathcal{L}$ is spanned by one polynomial $2(x_1x_2 + x_1x_3 + x_2x_3 + x_1 + x_2 + x_3) + 3$. In the case, the intersection $\mathcal{F} \cap \mathcal{H}$ coincides with the Clebsch diagonal surface.

On the other hand, if $n = 4$, $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 1$, and a large integer $\alpha_4 \gg 1$, then $\dim \mathcal{L} \geq 5$. At the limit $\alpha_n \to \infty$ the intersection $\mathcal{F} \cap \mathcal{H}$ converges to the Fermat surface inside the coordinate hyperplane $x_4 = 0$. The corresponding linear space contains five linearly independent polynomials. Thus, the same holds true for all sufficiently large integers α_4.

The examples have been computed by means of the service MathPartner [6].

Let us define

$$\lambda(n) = \frac{n(n+1)}{2} + 1$$

that is the upper bound on $\dim \mathcal{L}$. In case $\alpha_0 = \alpha_1$, the section $\mathcal{F} \cap \mathcal{H}$ contains the point $(-1, 0, \ldots, 0)$. Thus, for all $n \geq 4$, if $\mathcal{F} \cap \mathcal{H}$ is not a cone, then there exists a rational parametrization $\eta : \mathbb{Q}^{n-2} \dashrightarrow \mathcal{F} \cap \mathcal{H}$ defined over $\mathbb{Q}$, cf. [3]. Let the point $(-1, 0, \ldots, 0)$ be the image of the locus of indeterminacy; η can be found in probabilistic polynomial time.

If the section $\mathcal{F} \cap \mathcal{H}$ contains a point over the field $\mathbb{K}$, then there exists a rational parametrization $\eta : \mathbb{K}^{n-2} \dashrightarrow \mathcal{F} \cap \mathcal{H}$ defined over $\mathbb{K}$. Let the initial $\mathbb{K}$-point be the image of the locus of indeterminacy.

Theorem 2. *Given a multiset of positive integers $\alpha_0, \ldots, \alpha_n$, where $n \geq 4$ and $\mathcal{F} \cap \mathcal{H}$ is not a cone, and a real $\varepsilon > 0$. Let us consider the multilinear polynomials $M[f, \eta(P^{(k)})]$ for random points $P^{(k)}$, where the index k runs the segment $1 \leq k \leq \lambda(n)$, and all coordinates of the points $P^{(k)}$ are independent and uniformly distributed on the set of integers from one to $\lceil 2^{poly(n)}/\varepsilon \rceil$. The probability of spanning the whole linear space $\mathcal{L}$ is at least $1 - \varepsilon$.*

Proof. It is based on the Schwartz–Zippel lemma [7]. □

Thus, in case $n \geq 4$ and $\mathcal{F} \cap \mathcal{H}$ is not a cone, a basis of the linear space $\mathcal{L}$ can be computed in probabilistic polynomial time. In this way, the algorithm tries to find a rational parametrization $\eta : \mathbb{K}^{n-2} \dashrightarrow \mathcal{F} \cap \mathcal{H}$; if it failed because $\mathcal{F} \cap \mathcal{H}$ is a cone, then there exists a singular point. To prove the instance of the set partition problem has no solution, it is sufficient to check whether a nonzero constant belongs to the linear space $\mathcal{L}$. Of course, the condition is not necessary. On the other hand, if the linear space $\mathcal{L}$ contains a linear polynomial, one can reduce the dimension of the initial task.

References

[1] A. Schrijver, *Theory of linear and integer programming*. John Wiley and Sons, 1986.

[2] S. Margulies, S. Onn, and D.V. Pasechnik, On the complexity of Hilbert refutations for partition. *J. Symbolic Comput.* 66, 70–83 (2015). DOI:10.1016/j.jsc.2013.06.005

[3] J. Kollár, Unirationality of cubic hypersurfaces. *J. Inst. Math. Jussieu* 1:3, 467–476 (2002). DOI:10.1017/S1474748002000117

[4] I. Polo-Blanco, J. Top, A remark on parameterizing nonsingular cubic surfaces. *Computer Aided Geometric Design* 26, 842–849 (2009). DOI:10.1016/j.cagd.2009.06.001

[5] I.V. Latkin, A.V. Seliverstov, Computational complexity of fragments of the theory of complex numbers (in Russian). *Bull. Univ. Karaganda. Ser. Math.* 1, 47–55 (2015).

[6] G.I. Malaschonok, New generation of symbolic computation systems (in Russian). *Tambov University Reports. Series: Natural and Technical Sciences* 21:6, 2026–2041 (2016). DOI:10.20310/1810-0198-2016-21-6-2026-2041

[7] J.T. Schwartz, Fast probabilistic algorithms for verification of polynomial identities. *J. ACM* 27:4, 701–717 (1980). DOI:10.1145/322217.322225

Alexandr V. Seliverstov
Institute for Information Transmission Problems of the Russian Academy of Sciences (Kharkevich Institute), Moscow, Russia
e-mail: `slvstv@iitp.ru`

Symmetries and apparent singularities for simplest Fuchsian equation

Sergei Slavyanov

Simplest Fuchsian second order equations with particular emphasize to the role of apparent singularities are presented. The limiting transfer from Heun equation to hypergeometric equation is studied. The relation to Painlevé equation is exposed. Matrix formulation of the problem is traced. The talk is closing a gap in previous publications of the author.

Sergei Slavyanov
Sankt-Petersburg State University
198506, Petergof, Botanicheskaya 3

e-mail: slav@ss2034.spb.edu

Groups and Types

Sergei Soloviev

We consider some groups that appear naturally in type theories.

In each type theory T a natural notion of isomorphism of types is defined (*cf.* [Di Cosmo 1995]). For each type A one may define the groupoid $Gr(A)$ whose objects are all types A' in T isomorphic to A. Morphisms of $Gr(A)$ are isomorphisms between such types.

For each A the group $Aut(A)$ of automorphisms (i.e., isomorphisms $A \to A$) is also defined.

There are some other groups structures naturally associated with A. We consider: (i) the group of permutations $\Sigma(A)$ that respect the isomorphisms relation $\sim$: $\forall \sigma \in \Sigma(A).(\sigma(A) \sim A)$; (ii) the groupoid $Gr_\Sigma(A) \subseteq Gr(A)$ whose objects are $A' \sim A$ such that $\exists \sigma \in \Sigma(A).(A' = \sigma(A))$ and morphisms are the same isomorphisms as in $Gr(A)$ (*i.e.*, it is a full subcategory of $Gr(A)$; (iii) other groups generated by composition of isomorphisms and permutations that respect the isomorphism of types.

The purpose of our work is the study of the connections between these algebraic structures. We consider in detail the simply typed lambda-calculus $\lambda^1\beta\eta$ and the second order system $\lambda^2\beta\eta$ (see [Di Cosmo 1995]), but look to some extent at other systems, e.g., systems with depenent types, with coproduct etc.

Theorem 1. For every finite group G there exists some type A in $\lambda^1\beta\eta$ such that the group $\Sigma(A)$ is isomorphic to G.

Theorem 2. Let A be some type in $\lambda^1\beta\eta$ and $\overline{\forall}.A$ its universal closure (the type in the second order calculus $\lambda^2\beta\eta$). Then the group of automorphisms $Aut(\overline{\forall}.A)$ (in $\lambda^2\beta\eta$) is isomorphic to the cartesian product $Aut(A) \times \Sigma(A)$.

From these two theorems we derive the corollary that:

Corollary. For every finite group G there exists some type A in $\lambda^1\beta\eta$ such that the group $Aut(\overline{\forall}.A)$ (in $\lambda^2\beta\eta$) is isomorphic to G.

References

[Di Cosmo 1995] Di Cosmo, R. (1995) *Isomorphisms of types: from lambda-calculus to information retrieval and language design.* Birkhauser.

Sergei Soloviev
IRIT, University of Toulouse-3,
118, route de Narbonne,
31062, Toulouse, France,
e-mail: soloviev@irit.fr

On the Restriction of Smooth Plane Sextics

Tadashi Takahashi

Abstract. A double cover of the projective plane branched along a non-singular sextic curve is a genus 2 $K3$ surface. In this regard, the restriction of parameters of defining equation of non-singular sextic curve is an interesting object. We consider about the restriction of the defining equation that the dimension of the parameter is equal to 19.

1. Introduction

1.1. Sextic curves

Let $\mathbb{P}^2$ be a 2-dimensional complex projective space with the coordinate $[x, y, z]$ and let $f_6(x, y, z)$ be a homogeneous polynomial of variables x, y, z with degree 6 in $\mathbb{P}^2$. We consider the set

$$V_6 := \{(x, y, z) | f_6(x, y, z) = 0\}.$$

We call V_6 complex projective plane degree 6 curves (later, we call them sextic curves). The dimension of parameters of defining equation of non-singular sextic curve is less than or equal to 19 ([4]).

1.2. Singular point

Let $f(x, y, z)$ be a homogeneous polynomial of variables x, y, z with degree 6 in $\mathbb{C}^3$. Then

$$f(0,0,0) = \frac{\partial f(0,0,0)}{\partial x} = \frac{\partial f(0,0,0)}{\partial y} = \frac{\partial f(0,0,0)}{\partial z} = 0.$$

Hence, the analytic set defined by $f(x, y, z) = 0$ has a singular point at the origin in $\mathbb{C}^3$. The analytic set is a non-singular sextic curve in $\mathbb{P}^2$ if it has only isolated singular point at the origin in $\mathbb{C}^3$ (later, we call them smooth plane sextics). For a defining equation of an analytic set which has an isolated singular point at the origin in $\mathbb{C}^3$, there exist the following theorem([1]).

Theorem 1.1 (Arnold). *Let $f(z_0, z_1, z_2)$ be a polynomial in $\mathbb{C}^3$ and let V be an anlytic set such that $V = \{(z_0, z_1, z_2) | f(z_0, z_1, z_2) = 0\}$ which has an isolated singular point at the origin in $\mathbb{C}^3$. Then, for any i $(i = 0, 1, 2)$, there exists an integer $a_i \geq 1$ and f has a term $z_i^{a_i} z_j$.*

2. Elimination ideal and restriction

Theorem 2.1 (The Elimination Theorem [3]). *Let $I \subset k[x_1, \ldots, x_n]$ be an ideal and let G be a Gröbner basis of I with respect to the lex order where $x_1 > x_2 > \ldots > x_n$. Then, for every $0 \leq l \leq n$, the set*

$$G_l = G \cap k[x_{l+1}, \ \ldots \ , x_n]$$

is a Gröbner basis of the l-th elimination ideal I_l.

Theorem 2.2 (The Extension Theorem [3]). *Let $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$ and let I_l be the first elimination ideal of I. For each $1 \leq i \leq s$, write f_i in the form*

$$f_i = g_i(x_2, \ldots, x_n)x_1^{N_i} + \textit{terms in which } x_i \textit{ has degree} < N_i \ ,$$

where $N_i \geq 0$ and $g_i \in \mathbb{C}[x_2, \ldots, x_n]$ is nonzero. Suppose that we have a partial solution $(a_2, \ldots, a_n) \in V(I_l)$. If $(a_2, \ldots, a_n) \notin V(g_1, \ldots, g_s)$, then there exists $a_1 \in \mathbb{C}$ such that $(a_1, a_2, \ldots, a_n) \in V(I)$.

Let f be a defining equation, $I := \langle \ f, \dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \ldots, \dfrac{\partial f}{\partial x_n} \ \rangle$. Then we can obtain the non-degeneracy condition for the singularity at the origin from the Gröbner basis. We call this non-degeneracy condition the restriction([1]). We need to classify the conditions for the parameters of the leading term. We consider the ideal $I = \{f_i(t_1, \ldots, t_m, x_1, \ldots, x_n) : 1 \leq i \leq s\}$ in $k(t_1, \ldots, t_m)[x_1, \ldots, x_n]$ and fix a monomial order. Here, $t_1, \ldots, t_m$ are symbolic parameters appearing in the coefficients of $f_1, \ldots, f_s$. We can divide each f_i by its leading coefficient in $k(t_1, \ldots, t_m)$, under the assumption that these leading coefficients are equal to 1. Then let $g_1, \ldots, g_s$ be a reduced Gröbner basis for I, and the leading coefficients of g_i are then also 1 ([3]).

We consider a plane cubic in $\mathbb{C}^3$ as an example. Let $f = x^2z + y^3 + pyz^2 + qz^3$, then the set defined by $f = 0$ is a cubic curve in $\mathbb{C}^3$. If the cubic curve has an isolated singularity at the origin, the only solution of $(f, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}) = (0, 0, 0, 0)$ is $x = y = z = 0$. Let $f_1 = \dfrac{\partial f}{\partial x}$, $f_2 = \dfrac{\partial f}{\partial y}$, $f_3 = \dfrac{\partial f}{\partial z}$ and let G be the Gröbner basis of f, f_1, f_2, f_3. Then the Gröbner basis of the 3rd elimination ideal, I_3, is

$$G_3 = G \cap \mathbb{C}[p, \ q] = \phi.$$

And

$$G \cap \mathbb{C}[x, p, q] = x^3,$$
$$G \cap \mathbb{C}[y, p, q] = (4p^3 + 27q^2)y^4,$$
$$G \cap \mathbb{C}[z, p, q] = (4p^3 + 27q^2)z^4.$$

Hence, we obtain $4p^3 + 27q^2 = 0$ as the degeneracy condition of f. We consider the defining equation that has singularity and is constrained by the Milnor number of singularity. We call this condition the restriction of the definition equation's parameter, In this example, f is constrained by restriction $4p^3 + 27q^2 \neq 0$.

3. Restriction

We consider about only the defining equation that the dimension of the parameter is equal to 19. We got the restrictions on smooth plane quartics by using the Gr:obner basis. We use the method([5]). Then, for the smooth plane sextics, the following result holds.

Result 3.1. *In $\mathbb{P}^2$ with the coordinate $[x, y, z]$, let f be the defining equation that the dimension of the parameters of smooth plane sextics is equal to 19. Then f is as follows..*
$f = x^5z + (y^3 + a_1yz^2 + a_2z^3)x^3 + (a_3y^4 + a_4y^3z + a_5y^2z^2 + a_6yz^3 + a_7z^4)x^2$
$\quad + (a_8y^5 + a_9y^4z + a_{10}y^3z^2 + a_{11}y^2z^3 + a_{12}yz^4 + a_{13}z^5)x$
$\quad + a_{14}y^6 + a_{15}y^5z + a_{16}y^4z^2 + a_{17}y^3z^3 + a_{18}y^2z^4 + a_{19}yz^5 + a_{20}z^6$
where a_i is parametric coefficient ($1 \leq i \leq 20$).

The result of the calculation of the elimination ideal becomes the key to solve the problem of the restriction of the parameters.

Changing the coordinates so that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$.

Then the defining equation is as follows (We rewrite x', y', z' to x, y, z again after having transformed them) :
$f = x^5z + g_1x^4z^2 + (y^3 + g_2y^2z + g_3yz^2 + g_4z^3)x^3$
$\quad + +(g_5y^4 + g_6y^3z + g_7y^2z^2 + g_8yz^3 + g_9z^4)x^2$
$\quad + (g_{10}y^5 + g_{11}y^4z + g_{12}y^3z^2 + g_{13}y^2z^3 + g_{14}yz^4 + g_{15}z^5)x$
$\quad + g_{16}y^6 + g_{17}y^5z + g_{18}y^4z^2 + g_{19}y^3z^3 + g_{20}y^2z^4 + g_{21}yz^5 + g_{22}z^6$
where g_i is polynomial of α, β and a_i ($1 \leq i \leq 20$).

If $(g_{15}, g_{21}, g_{22}) = (0, 0, 0)$, then the curve defined by the defining equation has a singular point at [0,0,1] in $\mathbb{P}^2$.

We consider the existence of solution for the system of algebraic equations $(g_{15}, g_{21}, g_{22}) = (0, 0, 0)$. Here, we use elimination theorem(Theorem2.1).

Let I be the polynomial ideal $\langle g_{15}, g_{21}, g_{22}\rangle$ and let G be a Gröbner basis of I with respect to lexicographic order where $\alpha > \beta > a_1 > \ldots > a_{20}$. Then, the curve defined by the defining equation of result 3.1 has a singular point at [0,0,1] if and only if the Gröbner base of $G \cap \mathbb{C}[a_1, \ldots, a_{20}]$ is equal to 0.

We take their 2-th elimination ideals as follows:

$$G_2 \ = \ G \cap \mathbb{C}[a_1, \ldots, a_{20}].$$

Here, G_2 is a Gröbner basis of the 2-th elimination ideal I_2. Then G_2 consists of polynomials of $a_1, \ldots, a_{20}$. For any $g_i \in G_2$, $g_i = 0$ is a necessary condition to get the restriction of the defining equation that the dimension of the parameter is equal to 19.

And we must show the existence of the solution (α, β) in the system of algebraic equations $(g_{15}, g_{21}, g_{22}) = (0, 0, 0)$ for any value of the parameter coefficients.

If this Gröbner basis G_2 is obtained, we can get the restriction of the defining equation that the dimension of the parameter is equal to 19. However, we cannot carry out this calculation at large-scale algebra calculation.

This Gröbner basis G_2 may become the restriction of the defining equation that the dimension of the parameter is equal to 19. However, a concrete calculation is not yet possible. If the parameters are concrete numerical values, it can calculate. But the case except it is not possible.

So, we tried also the following methods. We use a resultant ([6]).

Let $R(g_{15}, g_{21})$ be the resultant of g_{15} and g_{21} in α. Similarly, let $R(g_{15}, g_{22})$ be the resultant of g_{15} and g_{22} in α.

$$R(g_{15}, g_{21}) = \sum_{i=0}^{23} c_i \beta^i, \quad R(g_{15}, g_{22}) = \sum_{i=0}^{27} c'_i \beta^i$$

where c_i, c'_i are polynomials of a_i ($1 \leq i \leq 20$).

Here, we put $h := 81R(g_{15}, g_{22}) - R(g_{15}, g_{21})\beta^4$. Then $h = \sum_{i=0}^{26} c''_i \beta^i$.

We consider the resultant of $R(g_{15}, g_{21})$ and h in β.

This resultant $\neq 0$ may become the necessary condition of the restriction of the defining equation that the dimension of the parameter is equal to 19. However, this method also does not enable the final calculation either.

References

[1] V. I. Arnol'd, Normal forms of functions in neighbourhoods of degenerate critical points, Russian Math. Surveys 29:2, pp. 10-50, 1974.

[2] B. Buchberger, Greobner Bases: an algorithmic method in polynomial ideal theory, in Multidimensional Systems Theory, edited by N. K. Bose. D. Reidel Publishing Company, Dordrecht, pp. 184-232, 1985.

[3] D. Cox, J. Little, D. O'Shea, Ideals, varieties, and algorithms - An introduction to computational algebraic geometry and commutative algebra, Springer-Verlag, 1997.

[4] R. Hartshone, Graduate Text in Math., 52: Algebraic Geometry", Springer-Verlag, New York, 1977.

[5] Tadashi Takahashi, Normal forms of smooth plane quartics and their restrictions, ScienceAsia 42S, pp. 26-33, 2016.

[6] B. L. van der Waerden, Moderne Algebra I,II, zweite verbesserte Auflage, Berlin-Leipzig, 1940.

Tadashi Takahashi
Dept. of Intelligence and Informatics
Konan University
Kobe, Japan
e-mail: takahasi@konan-u.ac.jp

Relations between discrete and smooth Morse theories

A. Zhukova

Abstract. We work with discrete Morse theory, that is a discrete analogue of the classical Morse theory. It was developed by R. Forman [1]. This theory can be applied to any simplicial and regular CW-complexes and, although its definition is quite simple, many classical results analogous to the ones of the continuous Morse theory arise in its scope: it allows to compute homologies, cup-product, Novikov homologies, develop Witten's deformation of the Laplacian, etc..

The classical construction of barycentric subdivision of simplicial complexes can be used to approximate a smooth structure on a triangulated topological manifold. We develop a simple algorithm to "transfer" a discrete Morse function, defined on a simplicial complex, onto the barycentric subdivision of this complex in such a way that all important data about this function (i. e. the number and dimensions of the critical simplexes and the gradient path structure) stays unchanged [2]. It can be done in several different ways (they differ inside the critical simplexes of the initial field) and so we can produce several different Morse functions.

References

[1] R. Forman, *Morse Theory for Cell Complexes*. Advances in Mathematics, 134, 90–145 1998.

[2] A. Zhukova, *Discrete Morse theory for the barycentric subdivision*, preprint (2016), available at `https://arxiv.org/abs/1605.04751`.

A. Zhukova
Faculty of Liberal Arts and Sciences
Saint-Petersburg State University
Saint-Petersburg, Russia
e-mail: `zhukova.a@spbu.ru`

Special and exceptional mock-Lie algebras (extended abstract)

Pasha Zusmanovich

Mock-Lie algebras are algebras satisfying two identities: commutativity

$$xy = yx$$

and the Jacobi identity

$$(xy)z + (zx)y + (yz)x = 0.$$

It is almost immediate that such algebras are Jordan algebras of nil index 3 (i.e., $x^3 = 0$). Conversely, over a field of characteristic $\neq 2, 3$, any Jordan algebra of nil index 3 is mock-Lie. Such algebras appeared in the literature under different names: Lie-Jordan, Jacobi-Jordan, "commutative" Lie algebras, etc. They live a dual life: as a very particular class of Jordan algebras, and as strange cousins of Lie algebras, and this work is an interesting blend of Lie and Jordan theories (and, of course, computer algebra).

The main question we are concerned with is which of those algebras admit a faithful representation, or, what is the same, admit embedding into an associative algebra (i.e., in Lie parlance, satisfy the Ado theorem, or, in Jordan parlance, are special).

The arguments used to establish the Ado theorem and the Poincaré–Birkhoff–Witt theorem – a fact closely related with the possibility of embedding of Lie algebras into associative ones – fail in somewhat curios ways in the mock-Lie case (one of those ways involves calculation of Gröbner bases of universal enveloping algebras of mock-Lie algebras).

Mock-Lie algebras of low dimension (≤ 6 at least) are special (i.e., embedded into an associative algebra). On the other hand, an exceptional (i.e., not special) mock-Lie algebra was constructed a long time ago in an unpublished preprint [1], and we reproduce these old efforts, showing how for any Jordan s-identity (Glennie, Thedy, Medvedev, etc.) one may produce a mock-Lie algebra which does not satisfy this identity, and hence is exceptional. The minimal dimension of so constructed algebra is 44.

Substantial computer calculations are involved, utilizing Albert [3] and GAP.

There are many open questions related to mock-Lie algebras: about minimal dimension and minimal degree of nilpotency of exceptional algebras, about cohomology theory, about mock-Lie and dual to it operads, etc.

Based on [2].

References

[1] I.R. Hentzel, D.P. Jacobs, and S.R. Sverchkov, *On exceptional nil of index* 3 *Jordan algebras*, Preprint, Novosibirsk State Univ., 1997.

[2] P. Zusmanovich, *Special and exceptional mock-Lie algebras*, Lin. Algebra Appl. **518** (2017), 79–96; arXiv:1608.05861.

[3] Albert; `http://www1.osu.cz/~zusmanovich/albert/`

Pasha Zusmanovich
Department of Mathematics
University of Ostrava
Ostrava, Czech Republic
e-mail: `pasha.zusmanovich@osu.cz`

VVM Publishing
St. Petersburg, ul. Shvetzova, 41.
Ord. 322.